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THE ELEMENTS

OF

ALGEBRA

DESIGNED FOR THE

USE OF STUDENTS IN THE UNIVERSITY,

BY THE LATE JAMES WOOD, D.D.,
DEAN OF ELY, AND
MASTER OF ST. JOHN'S COLLEGE, CAMBRIDGE.

BIBLIOTHECA BODLEIANA

TWELFTH EDITION,
CAREFULLY REVISED, AND ENLARGED
WITH
NOTES, ADDITIONAL PROPOSITIONS, AND
EXAMPLES,

BY THOMAS LUND, B.D.,
LATE FELLOW AND SADLERIAN LECTURER OF ST. JOHN'S COLLEGE, CAMBRIDGE.

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In this Edition a great portion of the supplementary matter, which before occupied the Appendices, is interspersed throughout the book, and is distinguished from Dr. Wood's text by being printed in a smaller type. Part II., on the General Theory of Equations, is entirely omitted, as it is a subject which does not now come before the student in his course of reading at so early a stage as it was formerly wont to do; and as modern discoveries have introduced a mode of treating it essentially different from that employed by Dr. Wood[*]. Other minor alterations have been made which it is not necessary to particularize: the whole work has been most carefully revised; and very great pains have been taken to ensure correctness of printing. Many additions also have been made to the Collection of Examples in the Appendix.

The Editor is again indebted for much valuable counsel and assistance to the Rev. L. P. Baker, Rector of Medbourn, formerly Fellow of St. John's College, as well as to many other of his friends engaged in tuition, for which he begs to offer his best thanks; and he will always be happy to receive suggestions for the improvement of the book from those whom practical experience enables to form a correct judgment upon any subject of which it treats.

The following document, lately received from the University Press, is inserted here, both for the sake of preserving it as a matter of history, and also to denote the uniform estimation in which Dr. Wood's work has been held for the long period of 50 years:—

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T. L.

Morton Rectory, March 1, 1845.

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[*] See Hymers' Treatise on the Theory of Equations.

[†] To the last 400 copies of this Edition Mr. Lund's Appendix was annexed.

"At the latter end of the last century Dr Wood, the present learned and venerable Master of St John's College, Cambridge, in conjunction with the late Professor Vince, undertook the publication of a series of elementary works on analysis, and on the application of mathematics to different branches of natural philosophy, principally with a view to the benefit of students at the Universities. The works of the latter of these two writers have already fallen into very general neglect, in consequence partly of their want of elegance, and partly in consequence of their total unfitness to teach the more modern and improved forms of those different branches of science. But the works of his colleague in this undertaking have continued to increase in circulation, and are likely to exercise for many years a considerable influence upon our national system of education; for they possess in a very eminent degree the great requisites of simplicity and elegance, both in their composition and in their design. The propositions are clearly stated and demonstrated, and are not incumbered with unnecessary explanations and illustrations. There is no attempt to bring prominently forward the peculiar views and researches of the author, and the different parts of the subjects discussed are made to bear a proper subordination to each other. It is the union of all these qualities which has given to his works, and particularly to his *Algebra*, so great a degree of popularity, and which has secured, and is likely to continue to secure, their adoption as text-books for lectures and instruction, notwithstanding the absence of very profound and philosophical views of the first principles, and their want of adaptation, in many important particulars, to the methods which have been followed by the great continental writers.

"In later times a great number of elementary works on Algebra, possessing various degrees of merit, have been published. Those, however, which have been written for purposes of instruction only, without any reference to the advancement of new views, either of the principles of the science, or to the extension of its applications, have generally failed in those great and essential requisites of simplicity, and of adequate, but not excessive, illustration, for which the work of Dr. Wood is so remarkably distinguished; whilst other works, which have possessed a more ambitious character, have been generally devoted too exclusively to the development of some peculiar views of their authors, and have consequently not been entitled to be generally adopted as text-books in a system of academical or national education."—*Professor Peacock's Report to the British Association* "*On Certain Branches of Analysis.*"

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CORRECTIONS.

Page	Line	Error	Correction.
97	10	2^m	$2m$.
151	4 from bottom	V	V_3.
174	4	$\left\{1+\frac{b^{\frac12}}{a^{\frac12}}\right\}$	$\left\{1+\frac{b^{\frac12}}{a^{\frac12}}\right\}^4$.
262	3	$\frac{1}{m}$	$\frac{1}{m^2}$.
viii	1	$a+b1$	$a+b$.

xi after line 11 add, (by reduction) $=\frac{(a-1-\sqrt{1+a^2})^2-3a}{a(3a-4)}$.

xii after line 4 add. (by reduction) $=\frac{1}{a}\{\sqrt{1+a^2}-1)(\sqrt{1-a^2}+1)\}^{\frac12}$.

INTRODUCTION.

VULGAR FRACTIONS.

ART. 1. A FRACTION is a quantity which represents a part or parts of an integer or whole.

2. A *simple fraction* consists of two members or *terms*, the *numerator* and the *denominator*; the denominator shews into how many equal parts the whole, or unity, is divided; and the numerator the number of those parts taken. The numerator is usually placed over the denominator with a line between them. Thus $\frac{2}{3}$, (two thirds,) signifies that unity is divided into three equal parts, and that two of those parts are taken.

It must be observed, that we suppose every integer to be divisible into any number of equal parts at pleasure.

3. A *proper fraction* is one whose numerator is less than its denominator, as $\frac{7}{8}$.

4. An *improper fraction* is one whose numerator is equal to, or greater than, its denominator, as $\frac{6}{6}$, $\frac{7}{5}$.

5. A *compound fraction* is a fraction of a fraction, as $\frac{3}{4}$ of $\frac{5}{6}$, where $\frac{5}{6}$ is the whole quantity of which $\frac{3}{4}$ is to be taken; also $\frac{2}{3}$ of $\frac{4}{5}$ of $\frac{9}{11}$ is a compound fraction; &c.

6. A quantity consisting of a whole number and a fraction is called a *mixed number*, as $7\frac{3}{10}$, which signifies 7 integers together with $\frac{3}{10}$ of an integer.

7. Every integer may be considered as a fraction whose denominator is 1; thus 5, or 5 units, is $\frac{5}{1}$.

8. A *continued fraction* is one whose denominator is continued by being itself a *mixed number*, and the denominator of the fractional part again continued as before, and so on: thus

$$\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}},\quad 7+\cfrac{5}{3+\cfrac{4}{9+\&c.}},$$ are called *continued fractions**.

9. *To multiply a fraction by any whole number multiply the numerator by that number and retain the same denominator.*

Thus $\frac{2}{15}$ multiplied by 7 is $\frac{14}{15}$. For the unit in each of the fractions $\frac{2}{15}$ and $\frac{14}{15}$ is divided into 15 equal parts, and 7 times as many of those parts are taken in the latter case as in the former.

10. *To divide a fraction by any whole number multiply the denominator by that number and retain the same numerator.*

Thus $\frac{3}{5}$ divided by 4 is $\frac{3}{20}$. For, the unit being divided into four times as many equal parts in $\frac{3}{20}$ as it is in $\frac{3}{5}$, each of the parts in the latter case is four times as great as in the former, and the same number of parts is taken in both cases; therefore the former fraction is one fourth of the latter.

11. *A simple fraction may be considered as representing the quotient arising from the division of the numerator by the denominator.*

---

* To avoid repetition the Reader is referred to the first section of the Algebra for the explanation of the signs +, −, ×, =, ÷, &c.

Thus the fraction $\frac{3}{4}$ represents the quotient of 3 divided by 4; for 3 is $\frac{3}{1}$ (Art. 7), and this divided by 4 is the fraction $\frac{3}{4}$ (Art. 10.) If the integer be supposed a pound, or twenty shillings, $\frac{3}{4}$ of £1, which is 15 shillings, is equal to $\frac{1}{4}$ of £3, which is also 15 shillings.

12. *If the numerator and denominator of a fraction be both multiplied by the same number, the value of the fraction is not altered.*

For, if the numerator be multiplied by any number, the fraction is multiplied by that number (Art. 9); and if the denominator be multiplied by the same number, the fraction is divided by it (Art. 10); and if a quantity be both multiplied and divided by the same number, its value is not altered.

$$\text{Thus } \frac{5}{14} = \frac{15}{42} = \frac{150}{420}, \text{ \&c.}$$

Cor. Hence, if the numerator and denominator of a fraction be both *divided* by the same number, its value is not altered; since $\frac{150}{420} = \frac{15}{42} = \frac{5}{14}$.

REDUCTION.

The operation by which a quantity is changed from one denomination to another, or by which a fraction has its *terms* diminished, *without altering its value*, is called *Reduction.*

13. *To reduce a whole number to a fraction with a given denominator.*

Multiply the proposed number by the given denominator- and the product will be the numerator of the fraction re, quired.

Ex. Reduce 5 to a fraction whose denominator is 6.

This is $\frac{5\times 6}{6}$ or $\frac{30}{6}$; because 5 may be considered as a fraction $\frac{5}{1}$ (Art. 7), the numerator and denominator of which are multiplied by 6, therefore its value is not altered. (Art. 12.)

14. *To reduce a mixed number to an improper fraction.*

Multiply the integral by the denominator of the fractional part, to this product add the numerator of the fractional part, and make its denominator the denominator of the sum.

Ex. 1. Reduce $7\frac{4}{5}$ to an improper fraction.

The quantity $7\frac{4}{5}$ is $7+\frac{4}{5}$, which is equal to $\frac{35}{5}+\frac{4}{5}=\frac{39}{5}$; for 7 (by last Art.) is equal to $\frac{35}{5}$, and if to this $\frac{4}{5}$ be added, the whole is $\frac{39}{5}$.

Ex. 2. Also $23\frac{9}{11}=\frac{253+9}{11}=\frac{262}{11}$.

15. *To reduce an improper fraction to a mixed number.*

Divide the numerator by the denominator for the integral part, and make the remainder the numerator of the fractional part, and the divisor its denominator.

Ex. Reduce $\frac{39}{5}$ to a mixed number.

The fraction $\frac{39}{5}=7\frac{4}{5}$; because the unit being divided into 5 parts, 39 such parts are to be taken, that is, 7 units and 4 such parts.

16. *To reduce a compound fraction to a simple one.*

Multiply all the numerators together for a new numerator, and all the denominators for a new denominator.

Ex. 1. $\frac{2}{3}$ of $\frac{4}{5} = \frac{8}{15}$; for one third of $\frac{4}{5}$ is $\frac{4}{15}$, (Art. 10); therefore two thirds of the same quantity, which must be twice as great, is $\frac{8}{15}$ (Art. 9).

Ex. 2. $\frac{3}{4}$ of $5 = \frac{3}{4}$ of $\frac{5}{1} = \frac{15}{4}$.

Mixed numbers must be reduced to improper fractions, before the rule can be applied. Thus

Ex. 3. $\frac{5}{8}$ of $\frac{2}{9}$ of $3\frac{1}{12} = \frac{10}{72}$ of $3\frac{1}{12} = \frac{10}{72}$ of $\frac{37}{12} = \frac{370}{864}$.

17. *To reduce a continued fraction to a simple one.*

Apply the rule (Art. 14) for reducing a mixed number to an improper fraction, commencing at the lowest extremity of the continued fraction, and proceeding gradually upwards until the whole is reduced to a simple fraction. But as this operation requires the use of a rule not yet proved, the example is deferred to Art. 38.

18. *To reduce a fraction to lower terms.*

Whenever the numerator and denominator of a fraction have a *common measure*, that is, *a number which divides each of them without remainder*, greater than unity, the fraction may be reduced to lower terms by dividing both the numerator and denominator by this common measure.

Ex. $\frac{105}{120}$ is reduced to $\frac{21}{24}$ by dividing both the numerator and denominator by 5; and $\frac{21}{24}$ is again reduced to $\frac{7}{8}$ by dividing its numerator and denominator by 3. That the value of the fraction is not altered appears from Art. 12.

In the same manner $\frac{168}{210} = \frac{84}{105} = \frac{28}{35} = \frac{4}{5}$.

19. *The "Greatest Common Measure" of two numbers is found by dividing the greater by the less, and the preceding divisor by the remainder, continually, till nothing is left: the last divisor is the greatest common measure required.*

Def. The Greatest Common Measure of two or more numbers is the *greatest* number which will divide each of them without remainder.

To find the greatest common measure of 189 and 224.

```
189) 224 (1
     189
     ---
      35) 189 (5
          175
          ---
           14) 35 (2
               28
               --
                7) 14 (2
                   14
                   --
                    0
```

By proceeding according to the rule, it appears that 7 is the last divisor, or the Greatest Common Measure sought. The proof of this rule will be given hereafter. See Art. 103.

20. *A fraction is reduced to its* lowest *terms by dividing its numerator and denominator by their* greatest *Common Measure.*

Ex. To reduce $\frac{385}{396}$ to its lowest terms.

By the Rule given in the last Art. the greatest Common Measure of the numerator and denominator is found to be 11; and therefore $\frac{35}{36}$ is the fraction in its lowest terms.

Cor. If unity be the greatest Common Measure of the numerator and denominator, the fraction is already in its lowest terms.

21. *To reduce any number of fractions to a common denominator.*

Having reduced, if there be any, compound fractions to simple ones, and mixed numbers to improper fractions, multiply each numerator by all the denominators except its own, for the new numerator, and all the denominators together for a common denominator.

Ex. 1. Reduce $\frac{1}{2}$, $\frac{2}{3}$ and $\frac{3}{4}$ to a common denominator.

$$\frac{1\times3\times4}{2\times3\times4},\ \frac{2\times2\times4}{2\times3\times4},\ \text{and}\ \frac{2\times3\times3}{2\times3\times4};\ \text{or}\ \frac{12}{24},\ \frac{16}{24}\ \text{and}\ \frac{18}{24}$$

are the fractions required. These fractions are respectively equal to the former, the numerator and denominator in each case having been multiplied by the same numbers, namely, the product of the denominators of the rest (Art. 12).

$$\frac{1\times3\times4}{2\times3\times4}=\frac{1}{2};\ \frac{2\times2\times4}{2\times3\times4}=\frac{2}{3};\ \text{and}\ \frac{2\times3\times3}{2\times3\times4}=\frac{3}{4}.$$

Ex. 2. Reduce $\frac{2}{5}$ of $\frac{3}{4}$ and $4\frac{1}{3}$ to a common denominator.

These are $\frac{6}{20}$ and $\frac{13}{3}$, or $\frac{3}{10}$ and $\frac{13}{3}$; therefore $\frac{9}{30}$ and $\frac{130}{30}$ are the fractions required.

22. *If the denominator of one of two fractions contain the denominator of the other a certain number of times exactly, multiply the numerator and denominator of the latter by that number, and it will be reduced to the same denominator with the former.*

Ex. Reduce $\frac{5}{12}$ and $\frac{2}{3}$ to a common denominator.

Since 12 contains 3 four times exactly, multiply both the numerator and denominator of $\frac{2}{3}$ by 4, and it becomes $\frac{8}{12}$, a fraction having the same denominator with $\frac{5}{12}$.

In the reduction of fractions to a common denominator the following rule is frequently required, in order that the reduced fractions may be *in their lowest terms:*—

23. *To find the "Least Common Multiple" of any numbers.*

Def. The "Least Common Multiple" of any numbers is the *least* number which is divisible by each of them without remainder.

To find it, write down in one line the numbers of which the least common multiple is required, separating them by some mark, as a comma. Divide *all those which have a common measure* by that common measure, and bring down the other numbers placed in a line with the quotients, separated as before; and repeat this process as long as any common measure exists between two or more of them. The Least Common Multiple required will be the continued product of the divisors and of the final quotients.

Ex. Required the Least Common Multiple of 8, 12, and 18.

$$\begin{array}{r|rrr} 2 & 8, & 12, & 18 \\ \hline 2 & 4, & 6, & 9 \\ \hline 3 & 2, & 3, & 9 \\ \hline & 2, & 1, & 3 \end{array}$$

Least Com. Mult. is $2\times2\times3\times2\times1\times3$ or 72.

The proof of this rule will be given hereafter. See Art. 115.

24. *To reduce fractions to a common denominator, in their lowest terms, find the "Least Common Multiple" of all the denominators, and make that the common denominator by multiplying both the numerator and denominator of each fraction by the quotient of Least Common Multiple divided by the denominator.*

Ex. Reduce to a common denominator $\frac{1}{3}$, $\frac{2}{7}$, $\frac{3}{14}$, $\frac{12}{21}$ and $\frac{3}{4}$.

The Least Common Multiple of the denominators by Art. 23 is found to be 84; and therefore the required fractions are

$$\frac{28\times1}{28\times3},\ \frac{12\times2}{12\times7},\ \frac{6\times3}{6\times14},\ \frac{4\times12}{4\times21},\ \frac{21\times3}{21\times4},$$

$$\text{or } \frac{28}{84},\ \frac{24}{84},\ \frac{18}{84},\ \frac{48}{84},\ \frac{63}{84}.$$

25. Cor. By reducing fractions to a common denominator their values may be compared.

Thus $\frac{4}{7}$ and $\frac{7}{12}$, when reduced to a common denominator, are $\frac{48}{84}$ and $\frac{49}{84}$; that is, the fractions have the same relative values that 48 and 49 have.

26. By reducing fractions to a common *numerator* also their values may be compared.

Thus $\frac{3}{13}$ and $\frac{4}{17}$, when reduced to a common numerator, are $\frac{12}{52}$ and $\frac{12}{51}$; and since the former of these fractions signifies that the unit is divided into 52 equal parts of which 12 are taken, and the latter signifies that the unit is divided into 51 equal parts of which 12 are taken, it is obvious that the latter fraction is the greater of the two, or that which has the smaller denominator.

27. *To find the value of a fraction of a proposed denomination in terms of a lower denomination.*

Multiply the fraction by the number of integers of the lower denomination contained in one integer of the higher, and the product is the value required. The value of any fractional part of the lower denomination may be obtained in the same manner, till we come to the lowest.

Ex. 1. What is the value of $\frac{5}{7}$ of a pound?

First, $\frac{5}{7}$ of £1 is $\frac{5}{7}$ of 20 shillings,

$$\text{or } \frac{5}{7} \text{ of } \frac{20}{1} \text{ shillings} = \frac{100}{7} = 14\tfrac{2}{7} \text{ shillings};$$

Next, $\frac{2}{7}$ of a shilling $= \frac{2}{7}$ of $\frac{12}{1}$ pence,

$$= \frac{24}{7} \text{ pence} = 3\tfrac{3}{7} \text{ pence};$$

Lastly, $\frac{3}{7}$ of a penny $= \frac{3}{7}$ of 4 farthings $= \frac{3}{7}$ of $\frac{4}{1}$,

$$\text{or } \frac{12}{7} \text{ farthings} = 1\tfrac{5}{7} \text{ farthings}:$$

$$\text{hence, } \frac{5}{7} \text{ of a pound is } \overset{s.}{14} \,.\, \overset{d.}{3} \,.\, \overset{q.}{1\tfrac{5}{7}}.$$

The operation is usually performed in the following manner:

£5
20
7) 100
14--2*s.*
12
7) 24
3--3*d.*
4
7) 12
1--5*q.*

Ans. $\overset{s.}{14}\ .\ \overset{d.}{3}\ .\ \overset{q.}{1\frac{5}{7}}$.

Ex. 2. What is the value of $\frac{5}{9}$ of a crown?

5*C*
5
9) 25
2--7*s.*
12
9) 84
9--3*d.*
4
9) 12
1--3*q.*

Ans. $\overset{s.}{2}\ .\ \overset{d.}{9}\ .\ \overset{q.}{1\frac{3}{9}}$.

28. *To reduce a quantity to a fraction of any denomination.*

Make the given quantity the numerator, and the number of integers of its denomination in one of the proposed denomination the denominator, and the fraction required is determined.

Ex. What fraction of a pound is $\overset{s.}{12} \,.\, \overset{d.}{7} \,.\, \overset{q.}{3}$?

$\overset{s.}{12} \,.\, \overset{d.}{7} \,.\, \overset{q.}{3} = \overset{q.}{607}$; and one pound $= \overset{q.}{960}$;

therefore $\frac{607}{960}$ is the fraction sought: because the integer being divided into 960 equal parts, $\overset{s.}{12} \,.\, \overset{d.}{7} \,.\, \overset{q.}{3}$ contains 607 such parts.

29. In the last example we were obliged to reduce the whole to farthings; and in general, if the higher denomination do not contain the lower an exact number of times, reduce them to a common denomination, and proceed as before.

Ex. What fraction of a guinea is half a crown?

Here sixpence is the greatest common denomination, of which a guinea contains 42, and half a crown 5, therefore $\frac{5}{42}$ is the fraction required.

Any common denomination would answer the purpose; but, if the greatest be taken, the resulting fraction is *in the lowest terms.*

30. *To reduce a fraction to any denomination.*

Find what fraction of the proposed denomination an integer of the denomination of the given fraction is, and the fraction required will be found by Art. 16.

Ex. 1. What fraction of a pound is $\frac{2}{3}$ of a shilling?

1 shilling is $\frac{1}{20}$ of a pound, therefore $\frac{2}{3}$ of 1 shilling is $\frac{2}{3}$ of $\frac{1}{20}$ of a pound, $= \frac{2}{60} = \frac{1}{30}$ of a pound.

Ex. 2. What fraction of a yard, is $\frac{5}{7}$ of an inch?

1 inch is $\frac{1}{36}$ of a yard, therefore $\frac{5}{7}$ of an inch is $\frac{5}{7}$ of $\frac{1}{36}$ of a yard, $= \frac{5}{252}$ of a yard.

Ex. 3. What fraction of a guinea is $\frac{4}{9}$ of a pound?

1 pound is $\frac{20}{21}$ of a guinea (Art. 29); hence $\frac{4}{9}$ of a pound is $\frac{4}{9}$ of $\frac{20}{21}$ of a guinea, $=\frac{80}{189}$ of a guinea.

ADDITION OF FRACTIONS.

31. *If fractions have a common denominator, their sum is found by taking the sum of the numerators, and subjoining the common denominator.*

Thus $\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$. For, if an integer be divided into five equal parts, one of those parts, together with two parts of the same kind, must make three such parts.

32. If the fractions have not a common denominator, reduce them to a common denominator, and proceed as before.

Ex. Required the sum of $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$.

These reduced to a common denominator are $\frac{40}{60}$, $\frac{45}{60}$ and $\frac{48}{60}$, whose sum is $\frac{133}{60}$, or $2\frac{13}{60}$.

33. When *mixed numbers* are to be added, to the sum of the fractional parts, found as before, add the sum of the integers.

Ex. Add together $5\frac{3}{4}$, $6\frac{1}{3}$ and $\frac{2}{5}$ of $\frac{1}{7}$.

$$\frac{3}{4}+\frac{1}{3}+\frac{2}{35}=\frac{315}{420}+\frac{140}{420}+\frac{24}{420}=\frac{479}{420}=1\tfrac{59}{420};$$

therefore the whole sum required is $5+6+1\frac{59}{420}$ or $12\frac{59}{420}$.

SUBTRACTION.

34. *The difference of two fractions which have a common denominator is found by taking the difference of their numerators, and subjoining the common denominator.*

Thus $\frac{4}{5} - \frac{3}{5} = \frac{1}{5}$. For, if the unit be supposed to be divided into five equal parts, and three of those parts be taken from four, the remainder must be one of the parts, or $\frac{1}{5}$.

35. If the fractions have not a common denominator, let them be reduced to a common denominator, and then take the difference as before.

Ex. 1. From $\frac{9}{11}$ take $\frac{4}{5}$.

$$\frac{9}{11} - \frac{4}{5} = \frac{45}{55} - \frac{44}{55} = \frac{1}{55}.$$

Ex. 2. From $\frac{11}{12}$ of $\frac{3}{5}$ take $\frac{1}{3}$ of $\frac{7}{8}$.

$$\frac{11}{12} \text{ of } \frac{3}{5} = \frac{33}{60}, \text{ and } \frac{1}{3} \text{ of } \frac{7}{8} = \frac{7}{24};$$

$$\frac{33}{60} - \frac{7}{24} = \frac{66}{120} - \frac{35}{120} \text{(Art. 24.)} = \frac{31}{120}.$$

When *mixed numbers* are to be subtracted, the integers may be subtracted separately, and then the fractional parts. Thus,

$$3\tfrac{1}{2} - 2\tfrac{1}{4} = 3 - 2 + \tfrac{1}{2} - \tfrac{1}{4} = 1 + \tfrac{1}{4} = 1\tfrac{1}{4}.$$

And if the fractional part of the mixed number to be subtracted is *greater* than that of the other, deduct a unit from the greater number and add it in a fractional form to the smaller fractional part; then proceed as before. Thus,

$$6\tfrac{1}{9} - 3\tfrac{7}{9} = 5\tfrac{10}{9} - 3\tfrac{7}{9} = 5 - 3 + \tfrac{10}{9} - \tfrac{7}{9} = 2 + \tfrac{3}{9} = 2\tfrac{1}{3}.$$

MULTIPLICATION.

36. Def. *To multiply one fraction by another is to take such part or parts of the former as the latter expresses.*

This is done by multiplying the numerators of the two fractions together for a new numerator, and the denominators for a new denominator.

Thus $\frac{3}{4} \times \frac{5}{7} = \frac{15}{28}$; for $\frac{3}{4}$ multiplied by $\frac{5}{7}$ is, according to the definition of multiplication, $\frac{5}{7}$ of $\frac{3}{4}$ or $\frac{15}{28}$, (Art. 16).

If there be more than two fractions to be multiplied together, a similar rule applies:—multiply all the numerators together for a new numerator, and all the denominators for a new denominator.

Thus, $\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{6}{24} = \frac{1}{4}$; for $\frac{1}{2}$ of $\frac{2}{3} = \frac{1}{3}$, and $\frac{1}{3}$ of $\frac{3}{4} = \frac{1}{4}$.

Compound fractions must be reduced to simple ones, and *mixed numbers* to improper fractions, and they may then be multiplied as before.

Ex. 1. Multiply $\frac{2}{5}$ of $\frac{9}{13}$ by $7\frac{1}{8}$.

$$\frac{2}{5} \text{ of } \frac{9}{13} = \frac{18}{65}; \text{ and } 7\tfrac{1}{8} = \frac{57}{8};$$

therefore their product is

$$\frac{18}{65} \times \frac{57}{8} = \frac{1026}{520} = 1\tfrac{506}{520} = 1\tfrac{253}{260}.$$

Ex. 2. Multiply $\frac{2}{217}$ by 7.

$$\frac{2}{217} \times 7 = \frac{2 \times 7}{217} = \frac{2}{31}.$$

Hence it appears, that a fraction may be multiplied by a whole number by dividing the denominator by that number, when this division can take place.

Much trouble is frequently saved by observing what multipliers are *common* to the new numerator and denominator, and striking them out (Art. 12) *before the multiplication is effected.*

Thus the product of $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{4}{5}$, is, by the rule,

$$\frac{1 \times 2 \times 3 \times 4}{2 \times 3 \times 4 \times 5},$$

which we see at once to be $\frac{1}{5}$, by striking out the quantity $2 \times 3 \times 4$ common to the numerator and denominator.

Again, if in the proposed fractions, to be multiplied together, there be *any* numerator and *any* denominator which have a common measure, divide them both by that common measure, and use the resulting quotients for the purpose of forming the required product.

Thus $\frac{18}{65} \times \frac{57}{8} = \frac{9}{65} \times \frac{57}{4} = \frac{513}{260}$; since 18 and 8 have the common measure 2.

Also $\frac{3}{4} \times \frac{7}{27} \times \frac{16}{21} = \frac{1}{4} \times \frac{7}{9} \times \frac{16}{21}$, (dividing 3 and 27 by 3,)

$= \frac{1}{1} \times \frac{7}{9} \times \frac{4}{21}$, (........ 4 and 16 by 4,)

$= \frac{1}{1} \times \frac{1}{9} \times \frac{4}{3}$, (........ 7 and 21 by 7,)

$= \frac{4}{27}$.

DIVISION.

37. *To divide one fraction by another, or to determine how often one is contained in the other, invert the numerator and denominator of the divisor, and proceed as in multiplication.*

Ex. 1. $\frac{3}{4}$ divided by $\frac{5}{7}$ is $\frac{3}{4} \times \frac{7}{5} = \frac{21}{20} = 1\frac{1}{20}$.

For, from the nature of division, the divisor multiplied by the quotient must produce the dividend: therefore $\frac{5}{7} \times$ quotient $= \frac{3}{4}$; let these equal quantities be multiplied by the same quantity $\frac{7}{5}$, and the products must be equal; that is, $\frac{7}{5} \times \frac{5}{7} \times$ quotient $= \frac{3}{4} \times \frac{7}{5}$, or $\frac{35}{35} \times$ quotient $= \frac{21}{20}$; but $\frac{35}{35} = 1$; therefore the quotient $= \frac{21}{20}$, as was found by the rule. And the same method of proof is applicable to all cases.

Compound fractions must be reduced to simple ones, and mixed numbers to improper fractions, before the rule can be applied.

Ex. 2. Divide $\frac{5}{9}$ of $\frac{4}{7}$ by $3\frac{1}{3}$.

$$\frac{5}{9} \text{ of } \frac{4}{7} = \frac{20}{63}, \text{ and } 3\tfrac{1}{3} = \frac{10}{3};$$

therefore the quotient required is $\frac{20}{63} \times \frac{3}{10} = \frac{2}{21}$.

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38. It will often happen in practice that fractions present themselves which require the application, not of one single rule *only*, as of Addition, or Subtraction, or Multiplication, &c., but of several rules in one operation. Thus,

Ex. 1. Required to find the single fraction which is equivalent to

$$\frac{5}{7} \times \left\{ 100 - \frac{2}{3} \text{ of } 100 + \frac{7\frac{1}{3}}{2\frac{1}{4}} \right\}.$$

First, $100 - \frac{2}{3}$ of $100 = \frac{1}{3}$ of $100 = \frac{100}{3}$,

$$\text{and } \frac{7\frac{1}{3}}{2\frac{1}{4}} = \frac{\frac{22}{3}}{\frac{9}{4}} = \frac{22}{3} \div \frac{9}{4} = \frac{88}{27};$$

therefore the whole quantity is equivalent to

$$\frac{5}{7} \times \left\{ \frac{100}{3} + \frac{88}{27} \right\}$$

$$= \frac{5}{7} \times \frac{900 + 88}{27}$$

$$= \frac{5}{7} \times \frac{988}{27} = \frac{4940}{189}$$

$$= 26\tfrac{26}{189}.$$

Ex. 2. Reduce $\dfrac{1}{2 + \dfrac{1}{3 + \dfrac{1}{4}}}$ to a simple fraction.

Here $3 + \frac{1}{4} = \frac{13}{4}$; and $\dfrac{1}{3 + \dfrac{1}{4}} = \dfrac{1}{\frac{13}{4}} = \dfrac{4}{13}$. Therefore

$$2 + \frac{1}{3 + \frac{1}{4}} = 2 + \frac{4}{13} = \frac{30}{13}, \text{ and the fraction required is } \frac{1}{\frac{30}{13}} \text{ or } \frac{13}{30}.$$

DECIMAL FRACTIONS.

39. In order to lessen the trouble which in many cases attends the use of vulgar fractions, decimal fractions have been introduced, which differ from the former in this respect, that their denominators are always 10 or some power of 10, as 100, 1000, 10000, &c. and instead of *writing* the denominator under the numerator, it is *expressed* by pointing off from the right of the numerator as many figures as there are cyphers in the denominator;

$$\text{thus } \cdot 2 \text{ signifies } \frac{2}{10}$$

$$\cdot 23 \;\ldots\ldots\; \frac{23}{100}$$

$$\cdot 127 \;\ldots\ldots\; \frac{127}{1000}$$

$$\cdot 0013 \;\ldots\ldots\; \frac{13}{10000}$$

$$43\cdot 7 \;\ldots\ldots\; 43\tfrac{7}{10} \text{ or } \frac{437}{10}.$$

40. Cor. 1. The value of each figure in a decimal decreases from the left to the right in a tenfold proportion, that is, each figure is ten times as great as if it were removed one place to the right, as in whole numbers; thus

$$\cdot 2 = \frac{2}{10}; \text{ but } \cdot 02 = \frac{2}{100}, \text{ and } \cdot 002 = \frac{2}{1000}, \text{ \&c.};$$

and the decimal ·127 is one tenth, two hundredths, and seven thousandths, of an unit.

41. Cor. 2. Adding cyphers to the right of a decimal does not alter its value; thus,

$$\begin{aligned} \cdot 2 = \frac{2}{10} &= \frac{20}{100} = \cdot 20 \\ &= \frac{200}{1000} = \cdot 200 \\ &= \frac{2000}{10000} = \cdot 2000 \\ &= \text{\&c.} \end{aligned}$$

the numerator and denominator having been multiplied by the same number. (Art. 12.)

42. Cor. 3. Decimals may be reduced to a common denominator by adding cyphers to the right, where it is necessary, till the number of decimal places is the same in all.

Ex. ·5, ·01, and ·311 reduced to a common denominator, are ·500, ·010 and ·311;

$$\text{that is } \frac{500}{1000}, \ \frac{10}{1000} \text{ and } \frac{311}{1000}.$$

43. Cor. 4. Hence in all complicated numerical reductions decimal fractions possess great advantages over vulgar fractions. For, 1st, the denominators of the former being always 10 or some power of 10, their reduction to a common denominator is easily effected, and consequently all operations requiring that previous reduction are facilitated: 2nd, the numerators and denominators of decimal fractions being usually written in one line, and the value of each figure decreasing in a tenfold proportion, from left to right, as in whole numbers, the common rules of Arithmetic are immediately applicable to such fractions, care only being taken, by means of rules for that purpose, to mark off correctly the decimal result.

As decimals are only fractions of a particular description, their operations must depend upon the principles already laid down.

ADDITION OF DECIMALS.

44. *To find the sum of any number of decimals place the figures in such a manner that those of the same denomination may stand under each other; add them together as in whole numbers, and place the decimal point in the sum under the other points.*

Ex. Add together 7·9, 51·43 and ·0118.

These, when reduced to a common denominator, are 7·9000 51·4300 and ·0118; and proceeding according to the rule,

$$\begin{array}{r} 7{\cdot}9000 \\ 51{\cdot}4300 \\ {\cdot}0118 \\ \hline 59{\cdot}3418 \\ \hline \end{array} = \text{the sum required. (Art. 31).}$$

In the operation the cyphers may be omitted, if the several decimal points stand exactly under each other, thus,

7·9
51·43
·0118

59·3418

SUBTRACTION.

45. *To find the difference of two decimals place the figures of the same denomination under each other; then subtract as in whole numbers, and place the decimal point under the other points.*

Ex. From 61·3 take 42·012.

These, reduced to a common denominator, are 61·300 and 42·012; therefore their difference is 19·288 (Art. 34). In the operation the cyphers may be omitted, thus,

61·3
42·012

19·288

MULTIPLICATION.

46. *To multiply one decimal by another multiply the figures as in whole numbers, and point off as many decimal places in the product as there are in the multiplier and multiplicand together.*

Ex. 51·3 × 4·6 = 235·98.

For $\frac{513}{10} \times \frac{46}{10} = \frac{23598}{100}$ = (according to the decimal notation) 235·98. And a similar proof may be given in all other cases.

47. When there are fewer figures in the product than there are decimals in the multiplier and multiplicand together, cyphers must be annexed to the *left* of the product, that the decimal places may be properly represented.

Ex. $\cdot 25 \times \cdot 3 = \cdot 075$; for $\frac{25}{100} \times \frac{3}{10} = \frac{75}{1000}$ = (according to the decimal notation) ·075.

DIVISION.

48. *Division in decimals is performed as in whole numbers, observing to point off as many decimals in the quotient as the number of decimal places in the dividend exceeds the number in the divisor.*

Ex. Divide 77·922 by 3·7.

$$\frac{77\cdot922}{3\cdot7} = 21\cdot06:$$

here there are three decimals in the dividend and one in the divisor; therefore, there are two in the quotient.

The truth of this rule is apparent from the nature of multiplication; for the product of the divisor and quotient is the dividend; there are, therefore, as many places of decimals in the dividend, as there are in the divisor and quotient together (Art. 46); consequently there are as many in the quotient as the number in the dividend exceeds the number in the divisor.

49. If figures be wanting in the quotient to make up the proper number of decimal places, cyphers must be added to the *left.*

Ex. Divide ·336 by 42.

$$\frac{336}{42} = 8;$$

and as the quotient of ·336 divided by 42 must contain three decimal places, that quotient is ·008.

For $\frac{336}{1000}$ divided by 42 is $\frac{336}{42000}$, or $\frac{8}{1000}$, that is (according to the decimal notation) ·008.

50. When the dividend does not contain as many decimals as the divisor, cyphers must be added to the *right* of the decimals in the dividend, till that is the case.

Ex. Divide 36 by ·012.

$$36 = 36\cdot000\,;$$

and 36·000 divided by ·012 is 3000, according to the rule.

REDUCTION.

51. *To reduce a vulgar fraction to a decimal.*

Add cyphers at pleasure, as decimals, in the numerator, and divide by the denominator according to the rule for the division of decimals. The truth of this rule is evident from Art. 11.

Ex. 1. $\frac{3}{4} = \frac{3\cdot00}{4} = \cdot75.$

Ex. 2. $\frac{7}{8} = \frac{7.000}{8} = \cdot875.$

Ex. 3. $\frac{4}{625} = \frac{4\cdot0000}{625} = \cdot0064.$

Ex. 4. $\frac{1}{3} = \frac{1\cdot000\ \&c.}{3} = \cdot333\ \&c.$

Ex. 5. $\frac{4}{33} = \frac{4\cdot0000\ \&c.}{33} = \cdot1212\ \&c.$

52. In some cases, as in the last two examples, the vulgar fraction cannot exactly be made up of tenths, hundredths, &c. but the decimal will go on without ever coming to an end*, *the*

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* It is evident that no vulgar fraction can be *exactly* expressed by a decimal, unless it either has, or can be reduced to another which has, 10 or some power of 10, for its denominator, (Art. 39). Thus, reverting to the Exs. of the last Art.

$$\frac{3}{4} = \frac{3\times5^2}{2^2\times5^2} = \frac{75}{10^2} = 0\cdot75,$$

$$\frac{7}{8} = \frac{7\times5^3}{2^3\times5^3} = \frac{875}{10^3} = 0\cdot875,$$

$$\frac{4}{625} = \frac{4\times2^4}{5^4\times2^4} = \frac{64}{10^4} = 0\cdot0064\,;$$

each of which vulgar fractions is expressed decimally with perfect exactness. But the following Exs. viz. $\frac{1}{3}$, and $\frac{4}{33}$, since they *cannot* be expressed by equivalent fractions with a denominator of 10 or some power of 10, are not capable of being expressed by terminating decimals.

Also since 10, and its powers, are divisible by 2 and 5 only, it follows that *no vulgar fraction can be expressed by a terminating decimal unless, when it is in its lowest terms, its denominator is divisible by one or both of the numbers,* 2 *and* 5, *and by no other number.*

*same figure or figures recurring in the same order**; but though we cannot represent the exact value of the vulgar fraction, yet, by increasing the number of decimal places, we may approach to it *as near as we please.* Thus $\frac{1}{9} = \cdot 1111$ &c. Now ·1, or $\frac{1}{10}$, is less than the true value by $\frac{1}{90}$; ·11, or $\frac{11}{100}$, is too little by $\frac{1}{900}$; and so on.

Again $\frac{41}{333} = \cdot 123123$ &c., the figures 123 being repeated without end; $\frac{4}{27} = \cdot 148148$ &c.; $\frac{5}{36} = \cdot 138888$ &c.; and so on.

Decimals of this kind are called *recurring or circulating* decimals.

Hence, although some vulgar fractions cannot be *accurately* represented by decimals, this affords no objection to the use of decimals, because for such fractions equivalent decimals can be found approximating to the true value *as nearly as we please*†.

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* This is easily shewn by a particular instance; and it may thence be seen to be true in all cases. Thus, suppose it is required to find the decimal equivalent to $\frac{4}{27}$. The required decimal is found by dividing 4·00000 &c. by 27; and if the quotient does not terminate, after each division there will be a remainder less than 27. Therefore, under the most unfavourable circumstances, at least after the quotient has reached to 26 figures, one of the remainders 1, 2, 3, &c....26 *must* recur; and consequently after that the figures in the quotient will recur. In the case proposed the remainder for one figure in the quotient is 13; for two figures, 22; for three figures, 4; which is a recurrence of the original figure: consequently the decimal is 0·148148, &c, the figures 148 being repeated *in infinitum.* Similarly also in other cases.

† "The addition, subtraction, multiplication, and division, of decimal fractions, are much easier than those of common fractions; and though we cannot reduce all common fractions to decimals, yet we can find decimal fractions so near to each of them, that the error arising from using the decimal instead of the common fraction will not be perceptible. For example, if we suppose an inch to be divided into ten million of equal parts, one of those parts by itself will not be visible to the eye. Therefore, in finding a length, an error of a ten-millionth part of an inch is of no consequence, even where the finest measurement is necessary."

"In applying Arithmetic to practice, nothing can be measured so accurately as to be represented in numbers without any error whatever, whether it be length, weight, or any other species of magnitude. It is therefore unnecessary to use any other than decimal fractions; since, by means of them, any quantity may be represented with as much correctness as by any other method." *De Morgan's Arithmetic,* 3rd Ed. 131.

53. The method of reducing a *terminating* decimal to a vulgar fraction is pointed out in Art. 39. The following method will serve for converting *recurring* decimals into their equivalent vulgar fractions.—

It appears, by actual division, that

$$\frac{1}{9} = 0{\cdot}11111 \;\ldots\ldots\ldots$$

$$\frac{1}{99} = 0{\cdot}010101 \;\ldots\ldots\ldots$$

$$\frac{1}{999} = 0{\cdot}001001 \;\ldots\ldots\ldots$$

$$\frac{1}{9999} = 0{\cdot}00010001 \;\ldots\ldots$$

and so on; where the recurring part of the decimal is always 1, preceded by as many ciphers as make the number of recurring digits equal to the number of 9's recurring in the denominator of the fraction.

If then, for instance, the vulgar fraction equivalent to 0·1212 &c. be required, we have

$$0{\cdot}1212 \text{ \&c.} = 0{\cdot}0101 \text{ \&c.} \times 12 = \frac{1}{99} \times 12 = \frac{12}{99} = \frac{4}{33}.$$

$$\text{Again } 0{\cdot}123123 \text{ \&c.} = 0{\cdot}001001 \text{ \&c.} \times 123 = \frac{1}{999} \times 123 = \frac{123}{999} = \frac{41}{333}.$$

If the recurring period does not begin with the first figure after the decimal point, multiply and divide by such a power of 10 as will move the decimal point to the required position; then proceed as before. Thus,

$$\text{Ex. 1.}\quad 0{\cdot}04545 \text{ \&c.} = \frac{0{\cdot}4545 \text{ \&c.}}{10} = \frac{1}{10} \times 0{\cdot}0101 \text{ \&c.} \times 45$$

$$= \frac{1}{10} \times \frac{45}{99} = \frac{45}{990} = \frac{1}{22}.$$

$$\text{Ex. 2.}\quad 0{\cdot}13888 \text{ \&c.} = \frac{13{\cdot}888 \text{ \&c.}}{100} = \frac{13}{100} + \frac{{\cdot}888 \text{ \&c.}}{100}.$$

$$\text{Now } 0{\cdot}888 \text{ \&c.} = 0{\cdot}111 \text{ \&c.} \times 8 = \frac{1}{9} \times 8 = \frac{8}{9};$$

$$\text{Therefore } 0{\cdot}13888 \text{ \&c.} = \frac{13}{100} + \frac{8}{900} = \frac{125}{900} = \frac{5}{36}.$$

Similarly it may be shewn, that $0{\cdot}09009009 \text{ \&c.} = \frac{10}{111}$.

54. If it be sufficient for the purposes of any calculation to take a number of decimals less than the number given or obtained, the following rule is to be observed:—

Rule. When the first of the figures struck off is 5 or >5, add 1 to the last remaining figure.

Thus, if 2·7182818 be the decimal under consideration, 2·72 is nearer to the true value than 2·71, for 2·7182818−2·71, is 0·0082818; and 2·72−2·7182818 is 0·0017182, which is considerably less than the former difference. Also 2·7183 is nearer to the true value than 2·7182, as may be shewn in a similar manner.

It may also be observed here, that in the multiplication of decimals some caution is requisite in taking the product as correct to a certain number of places of decimals, when either the multiplicand or multiplier is only approximately correct. Thus, if 3·12 express a certain length in inches, and is known to be correct within the thousandth part of an inch, the true length may be any thing between $3·12+\frac{1}{1000}$ and $3·12-\frac{1}{1000}$, that is, between 3·121 and 3·119; and if the proposed number is to be multiplied by 10, for example, the product is 31·2; whereas it may be any thing between 31·21 and 31·19, and therefore may not be correct even to one decimal place.

55. *To find the value of a decimal of one denomination in terms of a lower denomination.*

This may be done by the rule laid down in Art. 27.

Ex. Required the value of ·615625 £.

$$\begin{array}{rl}
·615625 & £ \\
20 & \\ \hline
12·312500 & \textit{shillings} \\
12 & \\ \hline
3·7500 & \textit{pence} \\
4 & \\ \hline
3·00 & \textit{farthings.} \\ \hline
\end{array}$$

The value required is $\overset{s.}{12} \;.\; \overset{d.}{3} \;.\; \overset{q.}{3}.$

First, ·615625 £ = 12·3125 *shillings.*

Next, ·3125 *s.* = 3·75... *pence.*

Lastly, ·75 *d.* = 3· ..... *farthings.*

56. *To reduce a quantity to a decimal of a superior denomination.*

Divide the quantity by the number of integers of its denomination contained in one of the superior denomination, and the quotient is the decimal required.

Ex. 1. What decimal of a shilling is threepence?

$$\begin{array}{r|l} 12 & 3{\cdot}00 \\ \hline & {\cdot}25 \ \textit{Ans.} \end{array}$$

For in the denomination shillings its numerical value must be $\frac{1}{12}$ of its value in the denomination pence.

Ex. 2. What decimal of a pound is $\overset{s.}{13}\ .\ \overset{d.}{4}\ .\ \overset{q.}{3}$?

$$\begin{array}{r|l} 4 & 3{\cdot}00 \\ \hline 12 & 4{\cdot}75 \\ \hline 20 & 13{\cdot}3958333 \ \&c. \\ \hline & {\cdot}66979166 \ \&c. \end{array}$$

First, we find what decimal of a penny $\overset{q.}{3}$ is; this, by the rule is ·75; then, what decimal of a shilling $\overset{d.}{4}\ .\ \overset{q.}{3}$ or 4·75*d.* is; this is found in the same manner to be ·3958333 &c.; lastly, we find, by the same rule, what decimal of a pound 13·3958333 &c. *sh.* is; which appears to be ·66979166 &c.

The conclusion will be the same if we reduce the quantity to a vulgar fraction (Art. 28), and this fraction to a decimal (Art. 51.)

57. It will often happen in practice that a whole *series* of vulgar fractions, instead of a single one, is to be reduced to a decimal, and in such cases considerable trouble may frequently be saved by making each fraction, when reduced, subservient to the reduction of some one or more of the others. Thus,

Ex. 1. *Required to reduce to a single decimal having* 5 *decimal places the following series of fractions:*—

$$\frac{2}{1}+\frac{1}{1\times2}+\frac{1}{1\times2\times3}+\frac{1}{1\times2\times3\times4}+\&c.$$

$$\begin{aligned}
\text{Here}\quad \frac{2}{1} & & &= 2{\cdot}000000 \\
\frac{1}{1\times2} &= \frac{1{\cdot}000}{2} & &= {\cdot}500000 \\
\frac{1}{1\times2\times3^{*}} &= \frac{{\cdot}5000}{3} & &= {\cdot}166667 \text{ (Art. 54.)} \\
\frac{1}{1\times2\times3\times4} &= \frac{{\cdot}166667}{4} & &= {\cdot}041667 \\
\frac{1}{1\times2\times3\times4\times5} &= \frac{{\cdot}041667}{5} & &= {\cdot}008333 \\
\frac{1}{1\times2\times3\times4\times5\times6} &= \frac{{\cdot}008333}{6} & &= {\cdot}001389 \\
\frac{1}{1\times2\times3\times4\times5\times6\times7} &= \frac{{\cdot}001389}{7} & &= {\cdot}000198 \\
\frac{1}{1\times2\times3\times4\times5\times6\times7\times8} &= \frac{{\cdot}000198}{8} & &= {\cdot}000025 \\
\frac{1}{1\times2\times3\times4\times5\times6\times7\times8\times9} &= \frac{{\cdot}000025}{9} & &= {\cdot}000003 \\
& & \text{Sum} &= 2{\cdot}718282
\end{aligned}$$

Hence the decimal required is 2·71828.

Each single fraction is calculated to *six* places of decimals, that the figure which occupies the 5th place in their sum may be correct; and no more terms of the series need be added, because the first five places of decimals are not affected by them.

Ex. 2. *Required to convert into a single decimal having five decimal places the following series:—*

$$2\times\left\{\frac{1}{7}+\frac{1}{3}\times\frac{1}{7^3}+\frac{1}{5}\times\frac{1}{7^5}+\frac{1}{7}\times\frac{1}{7^7}+\dots\right\}.$$

$$\begin{aligned}
\text{Here}\quad \frac{1}{7} &= 0{\cdot}1428571 \\
\frac{1}{7^2} &= \frac{0{\cdot}1428571}{7} = 0{\cdot}0204081 \\
\frac{1}{7^3} &= 0{\cdot}0029154 \\
\frac{1}{7^4} &= 0{\cdot}0004164
\end{aligned}$$

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* Some trouble in writing may be saved in this and similar examples by making the symbol $\underline{3|}$ stand for $1\times2\times3$, $\underline{4|}$ for $1\times2\times3\times4$, $\underline{5|}$ for $1\times2\times3\times4\times5$; and so on.

$$\frac{1}{7^5} = 0{\cdot}0000594$$

$$\frac{1}{7^6} = 0{\cdot}0000084$$

$$\frac{1}{7^7} = 0{\cdot}0000012.$$

Since 5 decimal places only are required, it is not necessary to add any more terms of the series. Therefore we have

$$\begin{array}{rr}
 & \frac{1}{7} = 0{\cdot}1428571 \\
 & \frac{1}{3} \times \frac{1}{7^3} = 0{\cdot}0009718 \\
 & \frac{1}{5} \times \frac{1}{7^5} = 0{\cdot}0000118 \\
 & \frac{1}{7} \times \frac{1}{7^7} = \underline{0{\cdot}0000001} \\
 & \text{Sum} = 0{\cdot}1438408 \\
\text{Mult. by} & \underline{2} \\
 & 0{\cdot}2876816;
\end{array}$$

Therefore the required decimal is 0·28768.

The proofs of the rules for the management of vulgar and decimal fractions here given are necessarily confined to particular instances, though the same reasoning may be applied in every case; and by using general signs, the proofs may be made general. But this requires a knowledge of Algebra.

EXAMPLES.

1. Reduce to *mixed numbers* $\frac{173}{18}$, $\frac{41}{40}$, and $\frac{2001}{200}$.
2. Reduce to *improper fractions* $9\frac{11}{18}$, $1\frac{1}{40}$, and $10\frac{1}{200}$.
3. What is $\frac{3}{4}$ of $\frac{5}{7}$ of $2\frac{1}{4}$? Ans. $1\frac{23}{112}$.
4. Reduce to lowest terms $\frac{5184}{6912}$, $\frac{7631}{26415}$, $\frac{236432}{2347432}$.

(1) Ans. $\frac{3}{4}$. (2) Ans. $\frac{13}{45}$. (3) Ans. $\frac{14}{139}$.

5. Reduce to a common denominator $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, and $\frac{1}{7}$.

Ans. $\frac{2520}{5040}$, $\frac{1680}{5040}$, $\frac{1260}{5040}$, $\frac{1008}{5040}$, $\frac{840}{5040}$, $\frac{720}{5040}$.

6. Reduce to a common denominator 21, $\frac{27}{16}$, $\frac{5}{8}$, and $1\frac{1}{2}$.

Ans. $\frac{336}{16}$, $\frac{27}{16}$, $\frac{10}{16}$, $\frac{24}{16}$.

7. Find the Least Common Mult. of 1, 2, 3, 4, 5, 6, 7, 8, and 9.
Ans. 2520.

8. Find the Least Common Mult. of 21, 22, 23, and 24.
Ans. 42504.

9. Find the Least Com. Mult. of 24, 7, 4, 21, and 14. Ans. 168.

10. Reduce to a common denominator $\frac{1}{7}$, $\frac{3}{8}$, $\frac{4}{9}$, $\frac{5}{24}$, $\frac{7}{72}$.

Ans. $\frac{72}{504}$, $\frac{189}{504}$, $\frac{224}{504}$, $\frac{105}{504}$, $\frac{49}{504}$.

11. What is the value of $\frac{3}{14}$ of half a guinea? Ans. 2*s*. 3*d*.

12. What is $\frac{2}{5}$ of a day $-\frac{3}{7}$ of an hour? Ans. 9*h*. $10\frac{2}{7}$*m*.

13. What fraction of half a crown is $\frac{3}{5}$ of 6*s*. 8*d*. Ans. $1\frac{3}{5}$.

14. Add together $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$. Ans. $3\frac{11}{20}$.

15. Add together $387\frac{1}{2}$, $285\frac{1}{4}$, $394\frac{1}{3}$, and $\frac{2}{5}$ of 3704. Ans. $2548\frac{41}{60}$.

16. From $201\frac{2}{3}$ take $97\frac{8}{9}$. Ans. $103\frac{7}{9}$.

17. Divide $1\frac{1}{2}$ by $1\frac{1}{4}$; and $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{10}$ by $\frac{1}{3}+\frac{1}{5}+\frac{1}{9}+\frac{1}{15}$.
(1) Ans. $1\frac{1}{5}$. (2) Ans. $1\frac{155}{256}$.

18. Convert 0·08, 0·00125, and 0·0078125 into their equivalent vulgar fractions. Ans. $\frac{2}{25}$, $\frac{1}{800}$, $\frac{1}{128}$.

19. Divide 3·1 by 0·0025; and 365 by 0·18349 to 6 places of decimals. (1) Ans. 1240. (2) Ans. 1989·209221.

20. Reduce 2*s*. $11\frac{3}{4}$*d*. to the decimal of £1. Ans. £0·14895833 &c.

21. Find the value of 0·07 of £2. 10*s*. and express the result as the decimal of £1. Ans. 3*s*. 6*d*. or £0·175.

22. Prove that 0·304565321 is more nearly represented by 0·30457 than by 0·30456.

THE
ELEMENTS OF ALGEBRA.

DEFINITIONS AND EXPLANATION OF SIGNS.

58. THE method of representing the relation of abstract quantities by letters and characters, which are made the signs of such quantities and their relations, is called ALGEBRA.

Algebra is the science of *generalization* as regards number and magnitude. Thus, for example, whilst Arithmetic teaches that the sum of the two numbers 6 and 4 multiplied by their difference is equal to the difference of their squares, Algebra teaches that the same is true for *any* two numbers whatever, whole or fractional.

59. Known or determined quantities are usually represented by the first letters of the alphabet, a, b, c, d, &c. and unknown or undetermined quantities, by the last, z, y, x, w, &c.

It must be observed, that this is simply a matter of agreement amongst Algebraical writers, for the sake of convenience, and not essential to the subject. Also, sometimes the letters of the Greek alphabet are used; sometimes A, B, C, D, &c. X, Y, Z; and sometimes others, according to circumstances or the will of the writer.

The following *signs* or *symbols* are made use of to express the relations which the quantities bear to each other:—

60. + (which is read *Plus*) signifies that the quantity to which it is prefixed must be added. Thus $a + b$ signifies that the quantity represented by b is to be added to the quantity represented by a; if a represent 5, and b represent 7, then $a + b$ represents 12. Also $a + b + c$ signifies that the *sum* of the quantities represented by a, b, and c, is to be taken.

If no sign be placed before a quantity, the sign + is understood: thus a signifies $+ a$. Such quantities are called *positive* quantities.

61. − (which is read *Minus*) signifies that the quantity to which it is prefixed must be subtracted. Thus $a - b$ signifies

that b must be taken from a; if a be 7, and b be 5, $a-b$ expresses 7 diminished by 5, or 2.

Quantities to which the sign $-$ is prefixed are called *negative* quantities.

$\pm$, or $\mp$, (the former of which is read *plus or minus*, the latter *minus or plus*) signifies that the quantity to which it is prefixed may be either added or subtracted. Thus 6 ± 4 is either 10 or 2.

62. $\times$ (which is read *Into*) signifies that the quantities between which it stands are to be multiplied together. Thus $a\times b$ signifies that the quantity represented by a is to be multiplied by the quantity represented by b*.

This sign is frequently omitted; thus abc signifies $a\times b\times c$. Or a full point is used instead of it; thus $1\times2\times3$, and $1.2.3$, signify the same thing.

But the sign must never be omitted, for obvious reasons, when two or more *numerals* are to be multiplied together.

Any quantity which, as a multiplier, serves to make up a product, is called a *factor* of that product. Thus, of the product $3\,abc$, each of the quantities 3, a, b, c is a *factor*; as also each of the quantities $3a$, $3b$, $3c$, $3ab$, $3ac$, $3bc$, ab, ac, bc, abc; the former being called *simple factors*, and the latter *compound factors*.

63. If in multiplication the same quantity be repeated *as a factor* any number of times, the product is usually expressed by placing, above the quantity, the number which represents how often it is repeated; thus a, $a\times a$, $a\times a\times a$, $a\times a\times a\times a$, &c. have respectively the same signification as a^1, a^2, a^3, a^4, &c. These quantities are called *powers*; thus a^1, or a, is called the first power of a; a^2 the second power, or square, of a; a^3 the third power, or cube, of a, &c. The numbers 1, 2, 3, &c. (thus affixed to a) are called the *indices* of a, or *exponents of the powers* of a.

Likewise a^1, a^2, a^3, &c. are said to be of one, two, three, &c. *dimensions* respectively; and, in general, any product is said to be of n *dimensions*, if the sum of the *indices* of its several *literal factors* is equal to n. Thus ab, that is a^1b^1 is of two dimensions; $3a^2b^3$ is of five dimensions; and so on.

64. $\div$ (which is read *Divided by*) signifies that the former of the quantities between which it is placed is to be divided by

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* By quantities we understand such magnitudes as can be represented by numbers; we may therefore without impropriety speak of the multiplication, division, &c. of quantities by each other.

the latter. Thus $a \div b$ signifies that the quantity a is to be divided by b.

The division of one quantity by another is frequently represented by placing the dividend over the divisor with a line between them, in which case the expression is called a *Fraction*. Thus $\frac{a}{b}$ signifies a divided by b (Art. 11); and a is the numerator, and b the denominator, of the fraction; also $\frac{a+b+c}{e+f+g}$ signifies that a, b, and c added together, are to be divided by e, f, and g added together.

65. A *power* in the denominator of a fraction is also expressed by placing it in the numerator, and prefixing the negative sign to its index; thus a^{-1}, a^{-2}, a^{-3}, a^{-n}, signify $\frac{1}{a^1}$, $\frac{1}{a^2}$, $\frac{1}{a^3}$, $\frac{1}{a^n}$ respectively; these are called the negative powers of a.

66. The sign $\sim$ between two quantities signifies their *difference*. Thus $a \sim x$, is $a - x$, or $x - a$, according as a or x is the greater; and $a \pm x$ signifies the sum or difference of a and x.

67. When several quantities are to be taken collectively, they are enclosed by *brackets*, as ( ), { }, [ ]. Thus $(a - b + c) \times (d - e)$ signifies that the quantity represented by $a - b + c$ is to be multiplied by the quantity represented by $d - e$.

Let a stand for 6; b, 5; c, 4; d, 3; and e, 1; then $a - b + c$ is $6 - 5 + 4$, or 5; and $d - e$ is $3 - 1$, or 2;

therefore $(a - b + c) \times (d - e)$ is 5×2, or 10.

Also $(ab - cd) \times (ab - cd)$ or $(ab - cd)^2$ signifies that the quantity represented by $ab - cd$ is to be multiplied by itself.

Sometimes a line, called a *vinculum*, is drawn over quantities, when taken collectively. Thus $\overline{a - b + c} \times \overline{d - e}$ means the same as $(a - b + c) \times (d - e)$.

68. = (which is read *Equals* or *is Equal to*) signifies that the quantities between which it is placed are equal to each other;

thus $ax - by = cd + ad$ signifies that the quantity $ax - by$ *is equal to* the quantity $cd + ad$.

69. The sign $>$ between two quantities signifies that the former *is greater than* the latter, and the sign $<$ that the former *is less than* the latter.

The sign $\therefore$ signifies *therefore,* and $\because$ *since* or *because.*

70. The *square root* of any proposed quantity is that quantity whose square, or second power, gives the proposed quantity. The *cube root* is that quantity whose cube gives the proposed quantity; and so on. The n^{th} root is that quantity whose n^{th} power gives the proposed quantity.

The signs $\sqrt{\ }$ or $\sqrt[2]{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, &c. $\sqrt[n]{\ }$ are used to express the square cube, biquadrate, &c. n^{th}, roots respectively of the quantities before which they are placed.

$$\sqrt[2]{a^2} = a, \quad \sqrt[3]{a^3} = a, \quad \sqrt[4]{a^4} = a, \quad \text{\&c.} \quad \sqrt[n]{a^n} = a.$$

These roots are also represented by the fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c. placed a little above the quantities, to the right. Thus $a^{\frac{1}{2}}$, $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, $a^{\frac{1}{n}}$, represent the square, cube, fourth, and n^{th}, root of a, respectively; $a^{\frac{5}{2}}$, $a^{\frac{7}{3}}$, $a^{\frac{3}{5}}$, represent the square root of the fifth power, the cube root of the seventh power, the fifth root of the cube, of a, respectively.

71. If any of these roots cannot be exactly determined, the quantities are said to be *irrational,* or are called *surds;* the quantities being called *rational,* when the roots expressed can be exactly determined.

72. Points are made use of to denote *proportion;* thus $a : b :: c : d$ signifies that a bears the same proportion to b that c bears to d.

73. The *number* prefixed to any quantity, *as a factor,* and which shews how often the quantity is to be taken, is called its *coefficient.* Thus, of $2a$, which signifies *twice* a, the coefficient is 2; and in the quantities $7ax$, $6by$, $3dz$, the numerals 7, 6 and 3 are called the coefficients of ax, by, and dz, respectively.

When no number is prefixed, the quantity is to be taken once, or the coefficient 1 is understood.

These numbers are sometimes represented by letters, which are also called coefficients.

Thus in the quantities px^3, qx^2, rx, we call p, q, and r the coefficients of x^3, x^2 and x respectively; since they may be read p times x^3, q times x^2, and r times x, respectively.

74. *Similar* or *like* algebraical quantities are such as differ only in their coefficients; $4a$, $6ab$, $9a^2$, $3a^2bc$, &c. are respectively similar to $15a$, $3ab$, $12a^2$, $15a^2bc$, &c.

Unlike quantities are different combinations of letters; thus ab, a^2b, abc, &c. are unlike.

But a distinction must be made in those cases where letters are taken to represent *coefficients*; for ax^3 and px^3 are *like* quantities when a and p are *coefficients* of x^3.

75. A quantity is said to be a *multiple* of another, when it contains it a certain number of times exactly; thus $16a$ is a multiple of $4a$, as it contains it exactly four times.

76. A quantity is called a *measure* of another, when the former is contained in the latter a certain number of times exactly; thus $4a$ is a measure of $16a$.

77. When two numbers have no common measure but unity, they are said to be *prime* to each other.

Thus 3 is *prime* to 7; 13 to 31; and so on.

A *prime number* is one which is prime to every other number.

78. A *simple* algebraical quantity is one which consists of a single term; as $4a$, or a^2bc, or $6xy$, &c.

A *compound* algebraical quantity consists of more terms than one—the number of *terms* meaning the number of quantities connected together by + or −; as $a+b$, or $2a-3x+4y$, &c.

A *binomial* is a quantity consisting of two terms, as $a+b$, or $2a-3bx$. A *trinomial* is a quantity consisting of three terms, as $2a+bd-3c$.

A *polynomial* or *multinomial* is a quantity consisting of many terms, as $a+bx+cx^2+dx^3+$ &c.

The following examples will serve to illustrate the method of representing quantities algebraically.

Let $a=8$, $b=7$, $c=6$, $d=5$ and $e=1$; then

Ex. 1. $3a - 2b + 4c - e = 24 - 14 + 24 - 1 = 33.$

Ex. 2. $ab + ce - bd = 56 + 6 - 35 = 27.$

Ex. 3. $\frac{a+b}{c-e} + \frac{3b-2c}{a-d} = \frac{8+7}{6-1} + \frac{21-12}{8-5} = \frac{15}{5} + \frac{9}{3} = 6.$

Ex. 4. $d^2 \times (a - c) - 3ce^2 + d^3 = 25 \times 2 - 18 + 125$

$= 50 - 18 + 125 = 157.$

Ex. 5. $\sqrt{a^2 - 3d} \times \sqrt[3]{b^3 - c^3 - 2e} = \sqrt{49} \times \sqrt[3]{125} = 7 \times 5 = 35.$

AXIOMS.

79. If equal quantities be added to equal quantities, the sums will be equal.

80. If equal quantities be taken from equal quantities, the remainders will be equal.

81. If equal quantities be multiplied by the same, or equal quantities, the products will be equal.

82. If equal quantities be divided by the same, or equal quantities, the quotients will be equal.

83. If the same quantity be added to and subtracted from another, the value of the latter will not be altered.

84. If a quantity be both multiplied and divided by another, its value will not be altered.

ADDITION OF ALGEBRAICAL QUANTITIES.

85. *The addition of algebraical quantities is performed by connecting those that are* unlike *with their proper signs, and collecting those that are* similar *into one sum.*

Add together the following *unlike* quantities;

Ex. 1. ax

$- by$

$+ e^2$

$- ed$

Sum $= \underline{ax - by + e^2 - ed}$

Ex. 2. $a + 2b - c$

$d - 5e + f$

Sum $= \underline{a + 2b - c + d - 5e + f}$

It is immaterial in what order the quantities are set down, if we take care to prefix to each its proper sign.

Generally speaking, however, it is convenient to arrange algebraical quantities in the order in which the letters occur in the alphabet.

When any terms are *similar*, they may be incorporated, and the general expression for the sum shortened.

1st. When *similar* quantities have the *same* sign, their sum is found by taking the sum of the coefficients with that sign, and annexing the common letters.

Ex. 3.
$$\begin{array}{rr} & 5a - 3b \\ & 4a - 7b \\ \hline \text{Sum} = & 9a - 10b \\ \hline \end{array}$$

Ex. 4.
$$\begin{array}{rr} & 4a^2c - 10bde \\ & 6a^2c - 9bde \\ & 11a^2c - 3bde \\ \hline \text{Sum} = & 21a^2c - 22bde \\ \hline \end{array}$$

The reason is evident; $5a$ to be added (Ex. 3), together with $4a$ to be added, makes $9a$ to be added; and $3b$ to be subtracted, together with $7b$ to be subtracted, is $10b$ to be subtracted.

2d. If *similar* quantities have *different* signs, their sum is found by taking the difference of the coefficients with the sign of the greater, and annexing the common letters as before.

Ex. 5.
$$\begin{array}{rr} & 7a + 3b \\ & -5a - 9b \\ \hline \text{Sum} = & 2a - 6b \\ \hline \end{array}$$

Ex. 6.
$$\begin{array}{rr} & \frac{4}{5}x - \frac{1}{2}y \\ & -\frac{1}{5}x + \frac{1}{4}y \\ \hline \text{Sum} = & \frac{3}{5}x - \frac{1}{4}y \\ \hline \end{array}$$

In the first part of the operation (Ex. 5) we have 7 times a to add, and 5 times a to take away; therefore upon the whole we have $2a$ to add. In the latter part, we have 3 times b to add, and 9 times b to take away; therefore we have upon the whole 6 times b to take away; and thus the sum of all the quantities is $2a - 6b$.

Ex. 7.
$$\begin{array}{rr} & a + b \\ & a - b \\ \hline \text{Sum} = & 2a \\ \hline \end{array}$$

Ex. 8.
$$\begin{array}{rr} & 1 - x + x^2 \\ & 1 + 2x - 2x^2 \\ \hline \text{Sum} = & 2 + x - x^2 \\ \hline \end{array}$$

It must be borne in mind that when any quantity as a, or x, or x^2, has no coefficient expressed, the coefficient 1 is understood.

3^d. If *several* similar quantities are to be added together, some with positive and some with negative signs, take the difference between the sum of the positive and the sum of the negative coefficients, prefix the sign of the greater sum, and annex the common letters.

$$\begin{array}{rl} \text{Ex. 9.} & 3a^2 + 4bc - e^2 + 10 \\ & -5a^2 + 6bc + 2e^2 - 15 \\ & -4a^2 - 9bc - 10e^2 + 21 \\ \hline \text{Sum} = & -6a^2 + bc - 9e^2 + 16 \end{array}$$

The method of reasoning in this case is the same as in Ex. 5.

$$\begin{array}{rl} \text{Ex. 10.} & 4ac - 15bd + ex \\ & 11ac + 7b^2 - 19ex \\ & -41a^2 + 6bd - 7de \\ \hline \text{Sum} = & 15ac - 41a^2 - 9bd + 7b^2 - 18ex - 7de \end{array}$$

$$\begin{array}{rl} \text{Ex. 11.} & px^3 - qx^2 - rx \\ & ax^3 - bx^2 - x \\ \hline \text{Sum} = & (p + a)\,x^3 - (q + b)\,x^2 - (r + 1)\,x \end{array}$$

In this example, letters are taken to represent coefficients, and the coefficients of like powers of x are enclosed within *brackets;* for it is evident that p times x^3 together with a times x^3 is the same as $(p+a)$ times x^3; also q times x^2 to be subtracted together with b times x^2 to be subtracted is the same as $(q+b)$ times x^2 to be subtracted; and r times x to be subtracted together with x, or $1x$, to be subtracted, is $(r+1)x$ to be subtracted.

SUBTRACTION.

86. *Subtraction, or the taking away of one quantity from another, is performed by changing the sign of the quantity to be subtracted, and then adding it to the other by the rules laid down in Art.* 85.

Ex. 1. From $2bx$ take cy, and the difference is properly represented by $2bx - cy$; because the $-$ prefixed to cy shews that it is to be subtracted from the other; and $2bx - cy$ is the sum of $2bx$ and $-cy$, Art. 85.

Ex. 2. Again, from $2bx$ take $-cy$, and the difference is $2bx + cy$; because $2bx = 2bx + cy - cy$, (Art. 83); take away $-cy$ from these equal quantities, and the differences will be equal (Art. 80); that is, the difference between $2bx$ and $-cy$ is $2bx + cy$, the quantity which arises from adding $+cy$ to $2bx$.

Ex. 3.
$$\begin{array}{ll} \text{From} & a + b \\ \text{take} & a - b \\ \hline \text{Difference} = & +2b \\ \hline \end{array}$$

Ex. 4.
$$\begin{array}{ll} \text{From} & 6a - 12b \\ \text{take} & -5a - 10b \\ \hline \text{Diff.} = & 11a - 2b \\ \hline \end{array}$$

Ex. 5.
$$\begin{array}{ll} \text{From} & 5a^2 + 4ab - 6xy \\ \text{take} & 11a^2 + 6ab - 4xy \\ \hline \text{Diff.} = & -6a^2 - 2ab - 2xy \\ \hline \end{array}$$

Ex. 6.
$$\begin{array}{ll} \text{From} & 4a - 3b + 6c - 11 \\ \text{take} & 10x + a - 15 - 2y \\ \hline \text{Diff.} = & -10x + 3a - 3b + 4 + 6c + 2y \\ \hline \end{array}$$

Ex. 7.
$$\begin{array}{ll} \text{From} & ax^3 - bx^2 + x \\ \text{take} & px^3 - qx^2 + rx \\ \hline \text{Diff.} = & (a - p)x^3 - (b - q)x^2 + (1 - r)x \\ \hline \end{array}$$

In this example the coefficients of like powers of x are *bracketed*, for reasons similar to those given in Ex. 10, Art. 85.

ADDITION AND SUBTRACTION BY BRACKETS.

In actual practice it seldom happens that either Addition or Subtraction of Algebraical quantities is presented to us as in the Examples, Arts. 85, and 86. All the quantities concerned are more commonly *in one line*, and are so retained through the whole operation, for the sake of convenience.

This arrangement renders necessary the frequent use of *Brackets*.

Thus Ex. 3. Art. 85 would stand $(5a - 3b) + (4a - 7b) = 9a - 10b$.

Ex. 3. Art. 86 .... $(a + b) - (a - b) = 2b$.

In the management of Brackets much care is needed, and the following rules are to be observed:—

87. Rule I. *If any number of quantities, enclosed by Brackets, be preceded by the sign* +, *the brackets may be struck out, as of no value or signification.*

Rule II. *If any number of quantities, enclosed by Brackets, be preceded by the sign* −, *the brackets and preceding sign may be struck out, if the signs of all the quantities within the brackets be changed, namely* + *into* −, *and* − *into* +.

Rule I. is obviously true; for in this case all that is meant is, that a number of quantities are to be added; and it can clearly make no difference whether they be added *collectively* or *separately*. Thus $a+(b+c)$ is equivalent to $a+b+c$; for the former signifies that the sum of b and c is to be added to a, which is evidently the sum of a, b, and c. Also $a+(b-c)$ is equivalent to $a+b-c$; for the former signifies that a quantity is to be added to a less than b by the quantity c; and the latter, that when b has been added to a, c must be subtracted, which is evidently the same thing.

Rule II. is proved thus:—

Let a, b, c represent any Algebraical quantities, simple or compound, of which $b+c$ is to be subtracted from a; this will be expressed by $a-(b+c)$.

Now if from a the portion b be taken, the result is $a-b$; but there is not enough subtracted from a by the quantity c, since $b+c$ was to be subtracted. Therefore c must also be subtracted, which leaves the result $a-b-c$; that is,

$$a-(b+c)=a-b-c.$$

Again, if $b-c$ is to be taken from a, this will be expressed by $a-(b-c)$.

Now if from a the quantity b be taken, the result is $a-b$, but there has been too much taken away by the quantity c, since $b-c$ only was to be subtracted; therefore c must be *added*, and the result becomes $a-b+c$; that is,

$$a-(b-c)=a-b+c.$$

The preceding rules apply also to quantities held together by a *vinculum;* since a *vinculum* serves the same purpose as *brackets*. Art. 67.

N.B. It is immaterial whether a vinculum or brackets be used in any case, that being dependent solely upon the will of the writer: but in some cases it is necessary, for distinction's sake, to use *both* at the same time, or else two kinds of brackets. Thus, to express a times the difference between b and $c-d$, we must write either $a\,.\,(b-\overline{c-d})$, or $a\,.\,\{b-(c-d)\}$. Consequently it requires to be especially noted, that in all cases when the Student meets with (or { or [, he must look, whatever may intervene, for the counterpart) or } or] respectively; and all that is included within the *complete bracket* must be treated, irrespective of other brackets or vincula, as the sign which precedes it directs. So that in striking out brackets by Rules I and II, each *pair* of symbols, as ( ), { }, [ ], must be struck out separately, and not all confusedly and at once.

A few examples will make this clearer.

Ex. 1. Perform the addition expressed by $(a+b)+(a-b)$.

$$\begin{aligned}(a+b)+(a-b)&=a+b+a-b, \text{ by Rule I,}\\ &=2a.\end{aligned}$$

Ex. 2. Perform the subtraction expressed by $(a+b)-(a-b)$.

$$\begin{aligned}(a+b)-(a-b)&=a+b-(a-b), \text{ by Rule I,}\\ &=a+b-a+b, \text{ by Rule II,}\\ &=2b.\end{aligned}$$

Ex. 3. Simplify $a-(x-a)-\{x-(a-x)\}$.

$$\begin{aligned}a-(x-a)-\{x-(a-x)\}&=a-x+a-x+(a-x)\\ &=a-x+a-x+a-x\\ &=3a-3x.\end{aligned}$$

Ex. 4. Simplify $1-\{1-(1-\overline{1-x})\}$.

$$\begin{aligned}1-\{1-(1-\overline{1-x})\}&=1-1+(1-\overline{1-x})\\ &=1-1+1-\overline{1-x}\\ &=1-1+1-1+x\\ &=x.\end{aligned}$$

Ex. 5. Simplify $-[-\{-(-a)\}]$.

$$\begin{aligned}-[-\{-(-a)\}]&=+\{-(-a)\}, \text{ by Rule II,}\\ &=-(-a), \text{ by Rule I,}\\ &=a, \text{ by Rule II.}\end{aligned}$$

MULTIPLICATION.

88. The multiplication of *simple* algebraical quantities must be represented according to the notation pointed out in Art. 62.

Thus $a\times b$, or ab, represents the product of a multiplied by b; abc the product of the three quantities a, b and c; and so on.

It is also indifferent in what order they are placed, $a\times b$ and $b\times a$ being equal.

For $1\times a=a\times 1$, or 1 taken a times is the same with a taken once; also b taken a times, or $b\times a$, is b times as great as 1 taken a times; and a taken b times, or $a\times b$, is b times as great as a taken once; therefore (Art. 81.) $b\times a=a\times b$. Also $abc=cab=bca=acb$; for, as in the former case, $1\times a\times b=a\times b\times 1$; and $c\times a\times b$ is c times as great as $1\times a\times b$; also $a\times b\times c$ is c times as great as $a\times b\times 1$; therefore $a\times b\times c=c\times a\times b$ (Art. 81); and a similar proof may be applied to the other cases.

89. To determine the *sign* of the product, observe the following rule:—

If the multiplier and multiplicand have the same *sign, the product is positive; if they have* different *signs, the product is negative.*

1st. $+a \times +b = +ab$; because in this case a is to be taken positively b times; therefore the product ab must be positive.

2d. $-a \times +b = -ab$; because $-a$ is to be taken b times; that is, we must take $-ab$.

3d. $+a \times -b = -ab$; for a quantity is said to be multiplied by a negative number $-b$, if it be subtracted b times; and a subtracted b times is $-ab$. This also appears from Art. 92. Ex. 2.

4th. $-a \times -b = +ab$. Here $-a$ is to be subtracted b times, that is, $-ab$ is to be subtracted; but subtracting $-ab$ is the same as adding $+ab$ (Art. 86); therefore we have to add $+ab$.

The 2d and 4th cases may be thus proved; $a - a = 0$, multiply these equals by b, then ab together with $-a \times b$ must be equal to $b \times 0$, or nothing*; therefore $-a$ multiplied by b must give $-ab$, a quantity which when added to ab makes the sum nothing.

Again, $a - a = 0$; multiply these equals by $-b$, then $-ab$ together with $-a \times -b$ must be equal to 0; therefore $-a \times -b = +ab$.

90. If the quantities to be multiplied have coefficients, these must be multiplied together as in common arithmetic; the sign and the literal product being determined by the preceding rules.

Thus $3a \times 5b = 15ab$; because $3 \times a \times 5 \times b = 3 \times 5 \times a \times b = 15ab$ (Art. 88).

Again, $4x \times -11y = -44xy$; $-9b \times -5c = +45bc$;

and $-6d \times 4m = -24md$.

91. *The powers of the same quantity are multiplied together by adding the indices;* thus $a^2 \times a^3 = a^5$, for $aa \times aaa = aaaaa$. In the same manner $a^6 \times a^{10} = a^{16}$; and $-3a^2x^3 \times 5axy^2 = -15a^3x^4y^2$.

---

* It is a common mistake of beginners to say that an algebraical expression which appears under the form $a \times 0$ is equal to a, by supposing it to signify a not multiplied at all; whereas, since $a \times b$ signifies a taken b times, in the same manner $a \times 0$ signifies a taken 0 times, and is therefore equal to 0.

To prove generally that $a^m \times a^n = a^{m+n}$, m *and* n *being any positive integers.*

By Def. Art. 63. $a^m = a \times a \times a \times a \times$ &c. continued to m factors;

$$\begin{aligned}
&\text{also} \quad a^n = a \times a \times a \times a \times \text{ \&c. } \ldots\ldots\ldots\ldots \; n \; \ldots\ldots; \\
&\therefore\; a^m \times a^n = a \times a \times a \ldots \text{to } m \text{ factors } \times a \times a \times a \ldots \text{to } n \text{ factors}, \\
&\qquad = a \times a \times a \ldots \text{to } \overline{m+n} \text{ factors}, \\
&\qquad = a^{m+n}, \text{ by Def. Art. 63.}
\end{aligned}$$

It will be *proved* hereafter (Art. 150) that the same rule holds when m and n are either fractional or negative.

Also $a^m b^n \times a^p b^q = a^m . a^p . b^n . b^q = a^{m+p} b^{n+q}$.

92. If the multiplier or multiplicand consist of several terms, each term of the latter must be multiplied by every term of the former, and the sum of all the products taken for the whole product of the two quantities.

$$\begin{array}{lll}
\text{Ex. 1.} & \text{Mult.} & a+b \\
& \text{by} & \underline{c+d} \\
& \text{Prod}^t. = & \underline{ac+bc+ad+bd}
\end{array}$$

Here $a+b$ is to be taken $c+d$ times, that is, c times and d times, or $(a+b)\,c+(a+b)\,d$.

$$\begin{array}{lll}
\text{Ex. 2.} & \text{Mult.} & a+b \\
& \text{by} & \underline{c-d} \\
& \text{Prod}^t. = & \underline{ac+bc-ad-bd}
\end{array}$$

Here $a+b$ is to be taken $c-d$ times, that is, c times wanting d times; or c times positively and d times negatively; that is, $(a+b)\,c-(a+b)\,d$, or $ac+bc-(ad+bd)$, or $ac+bc-ad-bd$, Art. 87.

$$\begin{array}{lll}
\text{Ex. 3.} & \text{Mult.} & a+b \\
& \text{by} & \underline{a+b} \\
\text{Prod}^t. \text{ by } a = & & a^2+ab \\
\ldots \text{ by } +b = & & \underline{+ab+b^2} \\
\text{Whole prod}^t. = & & \underline{a^2+2ab+b^2}
\end{array}$$

$$\begin{array}{lll}
\text{Ex. 4.} & \text{Mult.} & a+b \\
& \text{by} & \underline{a-b} \\
& & a^2+ab \\
& & \underline{-ab-b^2} \\
& \text{Prod}^t. = & \underline{a^2-b^2}
\end{array}$$

Ex. 5.
$$\begin{array}{ll}
\text{Mult.} & 3a^2 - 5bd \\
\text{by} & -5a^2 + 4bd \\
\hline
 & -15a^4 + 25a^2bd \\
 & \qquad\quad + 12a^2bd - 20b^2d^2 \\
\hline
\text{Prod}^{\text{t}}. = & -15a^4 + 37a^2bd - 20b^2d^2 \\
\hline
\end{array}$$

Ex. 6.
$$\begin{array}{ll}
\text{Mult.} & x + a \\
\text{by} & x + b \\
\hline
 & x^2 + ax \\
 & \quad + bx + ab \\
\hline
\text{Prod}^{\text{t}}. = & x^2 + (a + b)\,x + ab \\
\hline
\end{array}$$

Ex. 7.
$$\begin{array}{ll}
\text{Mult.} & a^2 + 2ab + b^2 \\
\text{by} & a^2 - 2ab + b^2 \\
\hline
 & a^4 + 2a^3b + a^2b^2 \\
 & \qquad - 2a^3b - 4a^2b^2 - 2ab^3 \\
 & \qquad\qquad + a^2b^2 + 2ab^3 + b^4 \\
\hline
\text{Prod}^{\text{t}}. = & a^4 - 2a^2b^2 + b^4 \\
\hline
\end{array}$$

Ex. 8.
$$\begin{array}{ll}
\text{Mult.} & x^m + x \\
\text{by} & x^n \\
\hline
\text{Prod}^{\text{t}}. = & x^{m+n} + x^{n+1} \\
\hline
\end{array}$$

Ex. 9.
$$\begin{array}{ll}
\text{Mult.} & 1 - x + x^2 - x^3 \\
\text{by} & 1 + x \\
\hline
 & 1 - x + x^2 - x^3 \\
 & \quad + x - x^2 + x^3 - x^4 \\
\hline
\text{Prod}^{\text{t}}. = & 1 - x^4 \\
\hline
\end{array}$$

Ex. 10.
$$\begin{array}{ll}
\text{Mult.} & \frac{1}{2}x - \frac{1}{3}y \\
\text{by} & 2x - 3y \\
\hline
 & x^2 - \frac{2}{3}xy \\
 & \quad - \frac{3}{2}xy + y^2 \\
\hline
\text{Prod}^{\text{t}}. = & x^2 - \frac{13}{6}xy + y^2 \\
\hline
\end{array}$$

Ex. 11.
$$\begin{array}{ll}
\text{Mult.} & x^2 - px + q \\
\text{by} & x + a \\
\hline
 & x^3 - px^2 + qx \\
 & \quad + ax^2 - apx + aq \\
\hline
\text{Prod}^{\text{t}}. = & x^3 - (p - a)\,x^2 + (q - ap)\,x + aq \\
\hline
\end{array}$$

Here the coefficients of x^2 and x are bracketed, since $-(p - a)\,x^2 = -px^2 + ax^2$; and $(q - ap)\,x = qx - apx$.

It may be useful to exhibit the Rules for Multiplication as follows:—

Rules. $p(a + b) = pa + pb$ .... (1),

$p(a - b) = pa - pb$ .... (2),

$$(a+b)(c+d)=ac+bc+ad+bd\ldots\ldots(3),$$
$$(a+b)(c-d)=ac+bc-ad-bd\ldots\ldots(4),$$
$$(a-b)(c-d)=ac-bc-ad+bd\ldots\ldots(5).$$

Assuming (1) and (2), which are too obvious to need a proof, to prove (3), let $a+b=m$, then,

$$\begin{aligned}(a+b)(c+d)&=m(c+d)\\&=mc+md,\text{ by (1)},\\&=(a+b)c+(a+b)d\\&=ac+bc+ad+bd,\text{ by (1)}.\end{aligned}$$

Similarly for (4).

To prove (5), let $a-b=m$, then

$$\begin{aligned}(a-b)(c-d)&=m(c-d)\\&=mc-md,\text{ by (2)},\\&=(a-b)c-(a-b)d\\&=ac-bc-(ad-bd)\\&=ac-bc-ad+bd,\text{ by Art. 87}.\end{aligned}$$

N.B. The Rules for the management of *Brackets*, given in Art. 87, apply only to the *Addition* and *Subtraction* of quantities so enclosed. If a collection of quantities within brackets is to be *multiplied* or *divided* by any quantity or collection of quantities, the brackets must not be struck out until the multiplication or division is actually performed. Thus $(a+b).(c+d)$ signifies that $a+b$ is to be taken $c+d$ times, and is obviously not the same as either $a+b(c+d)$, or $(a+b)c+d$. Again $(a+b)\div(c+d)$ is not equivalent to either $a+b\div(c+d)$, or $(a+b)\div c+d$; but it may be written $\frac{a+b}{c+d}$, the line which separates the numerator and denominator serving as a vinculum to both.

The learner would do well to practice multiplication of quantities by means of *brackets* as early as possible. Thus,

Ex. 1. $$\begin{aligned}(a-b)(c-d)&=(a-b)c-(a-b)d\\&=ac-bc-(ad-bd)\\&=ac-bc-ad+bd.\end{aligned}$$

Ex. 2. $$\begin{aligned}(x+a)(x+b)&=(x+a)x+(x+a)b\\&=x^2+ax+bx+ab\\&=x^2+(a+b)x+ab.\end{aligned}$$

Ex. 3. $$\begin{aligned}(x+1)(x+2)(x+3)&=(x^2+\overline{2+1}\,.\,x+2)(x+3).\quad\text{(Ex. 2.)}\\&=(x^2+3x+2)x+(x^2+3x+2)3\\&=x^3+3x^2+2x+3x^2+9x+6\\&=x^3+6x^2+11x+6.\end{aligned}$$

It is also useful *to commit to memory* at an early stage the three following results; as will appear from the subjoined Examples:—

$$(A+B).(A+B) \text{ or } (A+B)^2 = A^2 + B^2 + 2AB \ldots\ldots \text{(i)}$$
$$(A-B).(A-B) \text{ or } (A-B)^2 = A^2 + B^2 - 2AB \ldots\ldots \text{(ii)}$$
$$(A+B).(A-B) = A^2 - B^2 \ldots\ldots\ldots\ldots \text{(iii)}$$

whatever quantities A *and* B *may represent, simple or compound.*

Ex. 1. $(ax+by)(ax+by) = (ax)^2 + (by)^2 + 2.ax.by$, by (i),
$= a^2x^2 + b^2y^2 + 2abxy.$

Ex. 2. $(ax-by)(ax-by) = (ax)^2 + (by)^2 - 2.ax.by$, by (ii),
$= a^2x^2 + b^2y^2 - 2abxy.$

Ex. 3. $(ax+by)(ax-by) = (ax)^2 - (by)^2$, by (iii),
$= a^2x^2 - b^2y^2.$

Ex. 4. $(ax+b+cy+d)^2 = (\overline{ax+b} + \overline{cy+d})^2$
$= (ax+b)^2 + (cy+d)^2 + 2(ax+b)(cy+d)$
$= a^2x^2 + b^2 + 2abx + c^2y^2 + d^2 + 2cdy$
$+ 2acxy + 2adx + 2bcy + 2bd.$

Ex. 5. $(a+b+c)(a+b-c) = (\overline{a+b}+c)(\overline{a+b}-c)$
$= (a+b)^2 - c^2$, by (iii),
$= a^2 + b^2 + 2ab - c^2.$

Ex. 6. $(b+c-a)(c+a-b) = (c - \overline{a-b})(c + \overline{a-b})$
$= c^2 - (a-b)^2$, by (iii),
$= c^2 - (a^2 + b^2 - 2ab)$
$= c^2 - a^2 - b^2 + 2ab.$

SCHOLIUM.

The method of determining the sign of a product from the consideration of abstract quantities has been found fault with by some algebraical writers, who contend that $-a$, without reference to other quantities, is imaginary, and consequently not the object of reason or demonstration. In answer to this objection we may observe, that whenever we make use of the notation $-a$, and say it signifies a quantity to be subtracted, we make a tacit reference to other quantities.

Thus, in numbers, $-a$ represents a number to be subtracted from those with which it is connected; and when we suppose $-a$ to be taken b times, we must understand that a is to be taken b times from some other numbers. In estimating lines, or distances, $-a$ represents a line, or distance, in a particular direction. The negative sign does not render quantities imaginary, or impossible, but points out the relation of real quantities to others with which they are concerned.

DIVISION.

93. *To divide one quantity by another is to determine how often the latter is contained in the former, or what quantity multiplied by the latter will produce the former.*

Thus to divide ab by a is to determine how often a must be taken to make up ab, that is, what quantity multiplied by a will give ab; which we know is b.

From this consideration are derived all the rules for the division of algebraical quantities.

94. If the divisor and dividend be affected with *like* signs, the sign of the quotient is $+$; but if their signs be *unlike*, the sign of the quotient is $-$. Thus,

If $-ab$ be divided by $-a$, the quotient is $+b$; because $-a \times +b$ gives $-ab$; and a similar proof may be given in the other cases.

95. In the division of *simple* quantities, if the coefficient and literal product of the divisor be found in the dividend, the other part of the dividend, with the sign determined by the last rule, is the quotient.

Thus $-7b \div b = -7$; $-ax \div -a = x$; $14ab \div 7b = 2a . 7b \div 7b = 2a$.

Also $\dfrac{abc}{ab} = c$; because ab multiplied by c gives abc.

If we first divide by a, and then by b, the result will be the same; for $abc \div a = bc$, and $bc \div b = c$, as before.

96. Cor. Hence, *if any power of a quantity be divided by any other power of the same quantity, the quotient is the same quantity with an index which is found by taking the index of the divisor from the index of the dividend.*

For $\dfrac{a^5}{a^3} = \dfrac{a^3 \times a^2}{a^3}$ (Art. 91) $= a^2$; also $\dfrac{a^3}{a^6} = \dfrac{1}{a^3} = a^{-3}$ (Art. 65);

and generally, if m and n be positive integers, $\dfrac{a^m}{a^n} = a^{m-n}$; for

$$\frac{a^m}{a^n} = \frac{a \times a \times a \ldots \text{ to } m \text{ factors}}{a \times a \times a \ldots \text{ to } n \text{ factors}} = a \times a \times a \ldots \text{ to } \overline{m - n} \text{ factors, if } m > n$$

$$= a^{m-n}; \text{ or, if } m < n,\ \frac{a^m}{a^n} = \frac{1}{a^{n-m}} = a^{m-n} \text{ (Art. 65).}$$

Similarly $a^m b^n \div a^p b^q = a^{m-p} b^{n-q}$.

97. If only a part of the product which forms the divisor be contained in the dividend, the division must be represented by a fraction according to the direction in Art. 64, and the *factors* which are common to the divisor and dividend expunged.

Thus $15a^3b^2c \div -3a^2bx$, or $\dfrac{15a^3b^2c}{-3a^2bx} = \dfrac{-3a^2b \,.\, 5abc}{3a^2b \,.\, x} = \dfrac{-5abc}{x}$.

First, divide by $-3a^2b$, and the quotient is $-5abc$; this quantity is still to be divided by x (Art. 95), and as x is not contained in it, the division can only be represented in the usual way; that is, $\dfrac{-5abc}{x}$ is the quotient.

98. If the dividend consist of several terms, and the divisor be a simple quantity, every term of the dividend must be divided by it. Thus to divide $a^3x^2 - 5abx^3 + 6ax^4$ by ax^2,

$$\frac{a^3x^2 - 5abx^3 + 6ax^4}{ax^2} = \frac{a^3x^2}{ax^2} - \frac{5abx^3}{ax^2} + \frac{6ax^4}{ax^2} = a^2 - 5bx + 6x^2.$$

Also $$(a + b + c) \div abc = \frac{a}{abc} + \frac{b}{abc} + \frac{c}{abc}$$

$$= \frac{1}{bc} + \frac{1}{ac} + \frac{1}{ab}.$$

99. When the divisor consists of several terms, *arrange both the divisor and dividend according to the powers of some one letter* contained in them*, (that is, beginning with the highest power and going regularly down to the lowest, or *vice versâ*,) then find how often the first term of the divisor is contained in the first term of the dividend, and write down this quantity for the first term in the quotient; multiply the whole divisor by it, subtract the product from the dividend, and bring down to the remainder as

---

* The operation will be shortest when that letter is chosen whose highest power in the dividend comes nearest to the highest power of the same letter in the divisor.

many other terms of the dividend as the case may require, and repeat the operation till all the terms are brought down.

Ex. 1. If $a^2-2ab+b^2$ be divided by $a-b$, the operation will be as follows:

$$\begin{array}{l}
a-b)\ a^2-2ab+b^2\ (a-b \\
\qquad\quad a^2-ab \\
\hline
\qquad\qquad\ -ab+b^2 \\
\qquad\qquad\ -ab+b^2 \\
\hline
\qquad\qquad\quad *\quad *
\end{array}$$

The reason of this and the foregoing rule is, that as the whole dividend is made up of all its parts, the divisor is contained in the whole as often as it is contained in all the parts. In the preceding operation we inquire first, how often a is contained in a^2, which gives a for the first term of the quotient; then multiplying the whole divisor by it, we have a^2-ab to be subtracted from the dividend, and the remainder is $-ab+b^2$, with which we are to proceed as before.

The whole quantity $a^2-2ab+b^2$ is in reality divided into two parts by the process, each of which is divided by $a-b$; therefore the true quotient is obtained.

Ex. 2.

$$\begin{array}{l}
a+b)\ ac+ad+bc+bd\ (c+d \\
\qquad\quad ac+bc \\
\hline
\qquad\qquad\quad ad+bd \\
\qquad\qquad\quad ad+bd \\
\hline
\qquad\qquad\quad *\quad *
\end{array}$$

Ex. 3.

$$\begin{array}{l}
1-x)\ 1\quad (1+x+x^2+x^3+\&c.+\dfrac{\text{Remainder}}{1-x} \\
\qquad\ 1-x \\
\hline
\qquad\quad +x \\
\qquad\quad +x-x^2 \\
\hline
\qquad\qquad\ +x^2 \\
\qquad\qquad\ +x^2-x^3 \\
\hline
\qquad\qquad\qquad +x^3 \\
\qquad\qquad\qquad +x^3-x^4 \\
\hline
\qquad\qquad\qquad\quad +x^4\ \&c.
\end{array}$$

Ex. 4.

$$\begin{array}{l} y-1\,)\,y^3-1\,(\,y^2+y+1 \\ \qquad\;\underline{y^3-y^2\quad\;} \\ \qquad\quad +y^2-1 \\ \qquad\quad \underline{+y^2-y\quad} \\ \qquad\qquad\; +y-1 \\ \qquad\qquad\; \underline{+y-1} \\ \qquad\qquad\quad\;\; * \end{array}$$

From this example it appears that y^3-1 is divisible by $y-1$ without remainder, the quotient being y^2+y+1. It may be shewn in the same manner that x^3-a^3 is divisible by $x-a$, the quotient being x^2+ax+a^2; and that x^3+a^3 is divisible by $x+a$, the quotient being x^2-ax+a^2. These results are worth remembering.

Ex. 5. To divide $4ab^3+51a^2b^2+10a^4-48a^3b-15b^4$ by $4ab-5a^2+3b^2$.

First arrange the terms of both dividend and divisor according to the powers of a, beginning with the highest.

$$\begin{array}{l} -5a^2+4ab+3b^2\,)\,10a^4-48a^3b+51a^2b^2+4ab^3-15b^4\,(\,-2a^2+8ab-5b^2. \\ \qquad\qquad\qquad\quad\;\, \underline{10a^4-\;\,8a^3b-6a^2b^2\qquad} \\ \qquad\qquad\qquad\qquad\quad -40a^3b+57a^2b^2+4ab^3 \\ \qquad\qquad\qquad\qquad\quad \underline{-40a^3b+32a^2b^2+24ab^3\quad} \\ \qquad\qquad\qquad\qquad\qquad\qquad\; 25a^2b^2-20ab^3-15b^4 \\ \qquad\qquad\qquad\qquad\qquad\qquad\; \underline{25a^2b^2-20ab^3-15b^4} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\;\; * \qquad * \end{array}$$

Ex. 6.

$$\begin{array}{l} x-y\,)\;x^m-y^m\;(\,x^{m-1}+x^{m-2}y+x^{m-3}y^2\ldots+y^{m-1} \\ \qquad\quad \underline{x^m-x^{m-1}y\qquad} \\ \qquad\qquad +x^{m-1}y-y^m \\ \qquad\qquad \underline{+x^{m-1}y-x^{m-2}y^2\quad} \\ \qquad\qquad\qquad\quad +x^{m-2}y^2-y^m \\ \qquad\qquad\qquad\quad \underline{+x^{m-2}y^2-x^{m-3}y^3\quad} \\ \qquad\qquad\qquad\qquad\qquad +x^{m-3}y^3-y^m\ \&\text{c.} \end{array}$$

This division will obviously terminate without remainder for any proposed integral and positive value of m, when the quotient has reached to m terms, the last term being y^{m-1}. Hence we have,

$$(x^3-y^3)\div(x-y)=x^2+xy+y^2,$$
$$(x^4-y^4)\div(x-y)=x^3+x^2y+xy^2+y^3,$$
$$(x^5-y^5)\div(x-y)=x^4+x^3y+x^2y^2+xy^3+y^4;\ \text{and so on.}$$

Ex. 7.

$$\begin{array}{l}
x - a)\ x^3 - px^2 + qx - r\ (x^2 + (a-p)x + a^2 - pa + q \\
\qquad\quad x^3 - ax^2 \\
\hline
\qquad\quad (a-p)x^2 + qx \\
\qquad\quad (a-p)x^2 - (a^2 - pa)x \\
\hline
\qquad\qquad\qquad + (a^2 - pa + q)x - r \\
\qquad\qquad\qquad\ \ (a^2 - pa + q)x - (a^3 - pa^2 + qa) \\
\hline
\qquad\qquad\qquad\ \ \text{Remainder} \quad a^3 - pa^2 + qa - r \\
\hline
\end{array}$$

TRANSFORMATION OF FRACTIONS TO OTHERS OF EQUAL VALUE.

100. If the signs of all the terms both in the numerator and denominator of a fraction be changed, its value will not be altered.

For $\frac{-ab}{-a} = +b = \frac{+ab}{+a}$; and $\frac{ab}{-a} = -b = \frac{-ab}{+a}$.

Ex. $\frac{a-x}{b-x} = \frac{-a+x}{-b+x} = \frac{x-a}{x-b}$.

101. If the numerator and denominator of a fraction be both multiplied, or both divided, by the same quantity, its value is not altered.

For $\frac{ac}{bc} = \frac{a}{b}$ (Art. 97).

Hence a fraction is reduced to its lowest terms by dividing both the numerator and denominator by the greatest quantity that *measures* them both, which quantity is called the *Greatest Common Measure* of the numerator and denominator.

102. A fraction which has either its numerator or denominator a *simple* algebraical quantity is easily reduced to lowest terms; for the greatest common divisor of the numerator and denominator is at once found by inspection, that is, by observing at sight what factors are common. Thus to reduce $\frac{3a^2bc}{5a^3b^2d}$ to its lowest terms, we *see* that a^2b is the greatest common measure of the numerator and denominator, therefore the fraction required is $\frac{3c}{5abd}$.

Again, $\frac{2a+5ax}{7a}$ is at once reduced to $\frac{2+5x}{7}$; and $\frac{a^2}{a^2+ab}$ to $\frac{a}{a+b}$.

But in the case of fractions having for both numerator and denominator a *compound* algebraical quantity the following rule is often, though not always, needed.

103. *The Greatest Common Measure of two compound algebraical quantities is found by arranging them according to the powers of some letter (as in division), and then dividing the greater by the less, and the preceding divisor always by the last remainder, till the remainder is nothing; the last divisor is the greatest common measure required.*

Let a and b represent the two quantities, and let b be contained p times in a, with a remainder c; again, let c be contained q times in b, with a remainder d; and so on, till nothing remains; let d be the last divisor, and it will be the greatest common measure of a and b.

$$\begin{array}{l} b)\ a\ (p \\ \quad\underline{pb} \\ \quad c)\ b\ (q \\ \qquad\underline{qc} \\ \qquad d)\ c\ (r \\ \qquad\quad\underline{rd} \\ \qquad\quad\ 0 \end{array}$$

104. The truth of this rule depends upon these two principles;

1st. If one quantity measure another, it will also measure any multiple of that quantity. Let x measure y by the units in n, then it will measure my by the units in mn; for since $y = nx$, $my = mnx$ (Art. 81) $= mn\,.\,x$, that is, x is contained mn times in my, or measures my by the units in mn.

2d. If a quantity measure two others, it will measure their sum or difference. Let a be contained m times in x, and n times in y; then $ma=x$ and $na=y$; therefore $x \pm y = ma \pm na = (m \pm n)a$; that is, a is contained $m \pm n$ times in $x \pm y$, or it measures $x \pm y$ by the units in $m \pm n$.

105. Now it appears from the operation (Art. 103.) that $a - pb = c$, and $b - qc = d$; every quantity therefore, which measures a and b, measures pb, and $a - pb$, or c; hence also it measures qc, and $b - qc$, or d; that is, every common measure of a and b measures d.

It appears also from the division that $a = pb + c$, $b = qc + d$, $c = rd$; therefore d measures c, and qc, and $qc + d$ or b; hence it measures pb, and $pb + c$, or a. Every common measure then of *a* and *b* measures *d*, and *d* measures *a* and *b*; therefore *d* is their greatest common measure*.

Ex. 1. To find the Greatest Common Measure of a^2+2a+1 and a^3+2a^2+2a+1; and to reduce $\frac{a^2+2a+1}{a^3+2a^2+2a+1}$ to its lowest terms.

$$a^2+2a+1\big)\,a^3+2a^2+2a+1\,\big(a$$
$$\underline{a^3+2a^2+\ a}$$
$$a+1$$
$$a+1\big)\,a^2+2a+1\,\big(a+1$$
$$\underline{a^2+\ a}$$
$$a+1$$
$$\underline{a+1}$$
$$*\quad *$$

$a+1$ is therefore the Greatest Common Measure of the two quantities; and if they be respectively divided by it, the fraction is reduced to $\frac{a+1}{a^2+a+1}$, and is in its lowest terms.

Ex. 2. To find the Greatest Common Measure of a^4-x^4 and $a^3-a^2x-ax^2+x^3$; and to reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

$$a^3-a^2x-ax^2+x^3\big)\,a^4-x^4\,\big(a+x$$
$$\underline{a^4-a^3x-a^2x^2+ax^3}$$
$$a^3x+a^2x^2-ax^3-x^4$$
$$\underline{a^3x-a^2x^2-ax^3+x^4}$$
$$2a^2x^2-2x^4=2x^2(a^2-x^2)$$

leaving out $2x^2$, the next divisor is a^2-x^2.

$$a^2-x^2\big)\,a^3-a^2x-ax^2+x^3\,\big(a-x$$
$$\underline{a^3-ax^2}$$
$$-a^2x+x^3$$
$$\underline{-a^2x+x^3}$$
$$*$$

---

* This conclusion is more obvious when stated thus:—every common divisor of *a* and *b* is a divisor of *d*, but no quantity can be a divisor of *d* which is greater than *d*, therefore every common divisor of *a* and *b* is not greater than *d*; and since *d* is one of them, therefore *d* is the greatest.

Therefore $a^2 - x^2$ is the G. C. M. required; and the fraction is reduced to $\frac{a^2 + x^2}{a - x}$, and is in its lowest terms.

The quantity $2x^2$, found as a factor in every term of one of the divisors, $2a^2x^2 - 2x^4$, but not in every term of the dividend, $a^3 - a^2x - ax^2 + x^3$, must be left out; otherwise the quotient will be fractional, which is contrary to the supposition made in the proof of the rule: and by omitting this part, $2x^2$, no common measure of the divisor and dividend is left out, because, by the supposition, no part of $2x^2$ is found in all the terms of the dividend.

106. The proof of the rule for finding the G. C. M. of two algebraical quantities (Art. 103–5) excludes every case in which any one of the quotients p, q, r, is fractional; for mb is not a "multiple" of b, unless m be a *whole number*. But the fractional quotients may always be avoided by rejecting certain factors which are no part of the G. C. M. Thus the operation may be conducted as follows:

Required the G. C. M. of a and b.

$$\begin{array}{l}
b)\,a\,(p \\
\quad \underline{pb} \\
\quad c = mc' \text{ suppose} \\
c')\,b\,(q \\
\quad \underline{qc'} \\
\quad d = nd' \text{ suppose} \\
d')\,c'\,(r \\
\quad \underline{rd'} \\
\quad\ 0
\end{array}$$

neither m nor n being a "measure" of *both* a and b.

Now, d' is the G. C. M. required. For $a - pb = c = mc'$, and $b - qc' = d = nd'$. Every quantity, therefore, which measures a and b, measures pb, and $a - pb$ or mc', and therefore c', since m, by the supposition, contains no factor common to a and b; hence also it measures qc', and $b - qc'$ or nd'; and since n is no measure of a and b, therefore every common measure of a and b measures d'.

Again, since $a = pb + c$, $c = mc'$, $b = qc' + d$, $d = nd'$, $c' = rd'$; therefore d' measures c', and mc', and qc', and therefore $qc' + nd'$ or $qc' + d$ or b. Hence it measures pb and $pb + mc'$, and therefore $pb + c$ or a.

Every common measure, then, of a and b measures d', and d' measures a and b; therefore d' is the *greatest* common measure of a and b.

Sometimes the proposed quantities are of the form a, mb, or ma and b, in each of which cases, to avoid fractional quotients, the factor m is rejected, not being a part of the greatest common measure by supposition.

Similarly also, such a factor as m may be *introduced* into one of the quantities, if necessary.

The preceding proof may be easily modified to suit each of these cases.

From what has been said it will be seen that care must be taken, lest in rejecting a factor, we reject a part of the G. C. M.; as also in introducing a factor, lest we introduce one which will increase the common factors of the proposed quantities.

Ex. 1. Required the G.C.M. of $9x^3+53x^2-9x-18$ and $x^2+11x+30$.

$$\begin{array}{r}
x^2+11x+30)\ 9x^3+53x^2-9x-18\ (9x-46\\
\underline{9x^3+99x^2+270x}\\
-46x^2-279x-18\\
\underline{-46x^2-506x-1380}\\
227x+1362=227(x+6);
\end{array}$$

227 is evidently not a common divisor of the two proposed quantities, and may therefore be rejected.

$$\begin{array}{r}
x+6)\ x^2+11x+30\ (x+5\\
\underline{x^2+6x}\\
5x+30\\
\underline{5x+30}\\
*\quad *
\end{array}$$

$\therefore\ x+6$ is the G.C.M. required.

Ex. 2. Required the G.C.M. of $2x^3+x^2-8x+5$ and $7x^2-12x+5$.

To avoid a fractional quotient in the first step the former quantity must be multiplied by 7, which is not a factor of the other quantity, and will therefore not affect their G.C.M.

$$\begin{array}{r}
2x^3+\ x^2-\ 8x+\ 5\\
\underline{7}\\
7x^2-12x+5)\ 14x^3+\ 7x^2-56x+35\ (2x+4\\
\underline{14x^3-24x^2+10x}\\
31x^2-66x+35\\
\underline{28x^2-48x+20}\\
3x^2-18x+15=3(x^2-6x+5);
\end{array}$$

3 is evidently not a common divisor of the proposed quantities, and may therefore be rejected.

$$\begin{array}{r}
x^2-6x+5)\ 7x^2-12x+\ 5\ (7\\
\underline{7x^2-42x+35}\\
30x-30=30(x-1);\ \text{reject the factor } 30;
\end{array}$$

$$\begin{array}{rl}
x-1\,\big)\,x^2-6x+5\,\big(\,x-5 & \\
\underline{x^2-x} & \\
-5x+5 & \\
\underline{-5x+5} & \\
* \quad * &
\end{array}$$

$\therefore\ x-1$ is the G.C.M. required.

Ex. 3. Required the G.C.M. of $36a^6-18a^5-27a^4+9a^3$ and $27a^5b^2-18a^4b^2-9a^3b^2$.

Here $36a^6-18a^5-27a^4+9a^3=9a^3(4a^3-2a^2-3a+1)$,

and $27a^5b^2-18a^4b^2-9a^3b^2=9a^3b^2(3a^2-2a-1)$;

$\therefore\ 9a^3$ is a factor of the G.C.M., and the factor b^2 in the latter quantity may be rejected.

Proceeding with the other factors,

$$\begin{array}{rl}
 & 4a^3-2a^2-3a+1 \\
 & \underline{\phantom{4a^3-2a^2-3a+{}}3} \\
3a^2-2a-1\,\big) & 12a^3-6a^2-9a+3\,\big(\,4a \\
 & \underline{12a^3-8a^2-4a} \\
 & \phantom{12a^3-{}}2a^2-5a+3=2a(a-1)-3(a-1) \\
 & =(2a-3)(a-1);
\end{array}$$

$2a-3$ may be easily seen not to be a common divisor of the proposed quantities, and may therefore be rejected:

$$\begin{array}{rl}
a-1\,\big)\,3a^2-2a-1\,\big(\,3a+1 & \\
\underline{3a^2-3a} & \\
a-1 & \\
\underline{a-1} & \\
* \quad * &
\end{array}$$

$\therefore\ a-1$ is the G.C.M. of $4a^3-2a^2-3a+1$ and $3a^2-2a-1$.

Hence $9a^3(a-1)$ is the G.C.M. required.

107. To find the Greatest Common Measure of three quantities, a, b, c, find d the Greatest Common Measure of a and b; and the Greatest Common Measure of d and c is the Greatest Common Measure required.

Because every common measure of a, b, and c, measures d and c; and every measure of d and c measures a, b and c (Art. 105); therefore, the Greatest Common Measure of d and c must be the Greatest Common Measure of a, b and c.

108. In the same manner the Greatest Common Measure of four or more quantities may be found.

The Greatest Common Measure of four quantities, a, b, c, d, may be found by taking x the Greatest Common Measure of a and b, and y the Greatest Common Measure of c and d; then the Greatest Common Measure of x and y will be the common measure required.

That the continued division indicated in Art. 103 will come to an end is proved as follows:

109. If one quantity be divided by another, and the preceding divisor by the remainder, the remainder will at length be less than any quantity that can be assigned.

For $a = pb + c$; and b, and consequently pb, is greater than c; therefore $pb + c$, or a, is greater than $2c$, and $\frac{a}{2}$ is greater than c; therefore from a, a quantity greater than its half has been taken; in the same manner, when c is the dividend, more than its half is taken away, and so on; but if from any quantity there be taken more than its half, and from the remainder more than its half, and so on, there will, at length, remain a quantity less than any that can be assigned (Euc. x. 1).

110. In practice the Greatest Common Measure of two or more algebraical quantities is frequently found by a more expeditious method than the preceding, as follows. Taking Ex. 2. Art. 105.

Ex. Required the G.C.M. of $a^4 - x^4$ and $a^3 - a^2x - ax^2 + x^3$.

$$a^4 - x^4 = (a^2 + x^2)(a^2 - x^2).$$

Also $a^3 - a^2x - ax^2 + x^3 = a^2(a - x) - x^2(a - x)$

$$= (a^2 - x^2)(a - x);$$

therefore $a^2 - x^2$ is a common factor or divisor of the proposed quantities; and since the other factors $a^2 + x^2$ and $a - x$ have no common measure greater than 1, therefore $a^2 - x^2$ is the Greatest Common Measure required.

111. In the same manner fractions are usually reduced to their lowest terms without the application of the Rule for finding the G.C.M. of the numerator and denominator.

Ex. 1. Reduce to lowest terms $\frac{x^2 + 11x + 30}{9x^3 + 53x^2 - 9x - 18}$.

$$\frac{x^2 + 11x + 30}{9x^3 + 53x^2 - 9x - 18} = \frac{x(x + 6) + 5(x + 6)}{9x^2(x + 6) - (x^2 + 9x + 18)}$$

$$= \frac{(x+5)(x+6)}{9x^2(x+6)-(x+3)(x+6)}$$

$$= \frac{(x+5)(x+6)}{(9x^2-x-3)(x+6)}$$

divid^{g}. num^{r}. and denom^{r}. by $x+6 = \dfrac{x+5}{9x^2-x-3}$.

Ex. 2. Reduce $\dfrac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}$ to its lowest terms.

$$x^2+(a+c)x+ac = x^2+ax+cx+ac$$

$$= x(x+a)+c(x+a)$$

$$= (x+c)(x+a);$$

also $x^2+(b+c)x+bc = (x+c)(x+b)$;

$\therefore$ the fraction becomes $\dfrac{(x+a)(x+c)}{(x+b)(x+c)}$,

and in lowest terms is $\dfrac{x+a}{x+b}$.

Ex. 3. Reduce $\dfrac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}$ to its lowest terms.

$$\frac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2} = \frac{3a^2(a-b)+b^2(a-b)}{4a(a-b)-b(a-b)}$$

$$= \frac{(3a^2+b^2)(a-b)}{(4a-b)(a-b)}$$

$$= \frac{3a^2+b^2}{4a-b}.$$

Ex. 4. Reduce $\dfrac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$ to its lowest terms.

$$\frac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2} = \frac{(a+b)(a+b+c)(a+b-c)}{(2bc+a^2-b^2-c^2)(2bc-a^2+b^2+c^2)},$$

$$= \frac{(a+b)(a+b+c)(a+b-c)}{\{a^2-(b-c)^2\}\{(b+c)^2-a^2\}}$$

$$= \frac{(a+b)(a+b+c)(a+b-c)}{(a+b-c)(a+c-b)(a+b+c)(b+c-a)},$$

$$= \frac{a+b}{(a+c-b)(b+c-a)}$$

$$= \frac{a+b}{c^2-(a-b)^2}.$$

112. *Fractions may be changed to others of equal value, with a common denominator, by multiplying each numerator by every denominator except its own, for the new numerator, and all the denominators together for the common denominator.*

Let $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ be the proposed fractions; then $\frac{adf}{bdf}$, $\frac{cbf}{bdf}$, $\frac{edb}{bdf}$, are fractions of the same value respectively with the former, having the common denominator bdf. For $\frac{adf}{bdf} = \frac{a}{b}$; $\frac{cbf}{bdf} = \frac{c}{d}$; and $\frac{edb}{bdf} = \frac{e}{f}$ (Art. 101); the numerator and denominator of each fraction having been multiplied by the same quantity, *viz.* the product of the denominators of all the other fractions.

113. When the denominators of the proposed fractions are not *prime* to each other, find their Greatest Common Measure; multiply both the numerator and denominator of each fraction by the denominators of all the rest, divided respectively by their Greatest Common Measure; and the fractions will be reduced to a common denominator *in lower terms* than they would have been by proceeding according to the former rule.

Thus $\frac{a}{mx}$, $\frac{b}{my}$, $\frac{c}{mz}$, reduced to a common denominator, are $\frac{ayz}{mxyz}$, $\frac{bxz}{mxyz}$, $\frac{cxy}{mxyz}$.

114. To obtain them in the *lowest* terms each must be reduced to another of equal value, with the denominator which is the *Least Common Multiple* of all the denominators.

It becomes necessary therefore to investigate a rule for finding the Least Common Multiple of two or more quantities. And first,

115. *To find the Least Common Multiple of any number of* simple *quantities.*

To do this, observe the combinations of letters which form the several quantities, and resolve each quantity, by inspection, into its simple factors. The object then will be, to construct such a quantity as shall contain every factor found in all the proposed quantities, but no factor repeated which is not also similarly repeated in some one of them; for thus we shall obviously form a quantity, and the *least* quantity, which is divisible by each of the proposed quantities without remainder, that is, the Least

Common Multiple of them all. To this end, detach from each quantity all the factors, which are common to two or more of them, until the quantities are left prime to each other. The continued product of these common factors and prime results will be the Least Common Multiple required.

Thus, let the L.C.M. of $2a$, $6ab$, and $8ab$ be required. We see that the three quantities have a common factor $2a$, which being detached leaves the quantities 1, $3b$, and $4b$: of these again, the two latter have a common factor b, which being detached leaves the quantities 1, 3, and 4; and these are prime to each other. Therefore the L.C.M. required is

$$2a \times b \times 1 \times 3 \times 4 \text{ or } 24ab.$$

OBS. Since the detaching of the Common Factors is the same thing as dividing the quantities by their Greatest Common Measures, it is clear that this method coincides with the arithmetical rule given in Art. 23.

It may also be observed that the preceding method is applicable to *compound* quantities, as well as *simple*, provided that each of the quantities can be readily resolved into its component factors Thus, if the L.C.M. of $ab + ad$, and $ab - ad$ be required, we see that the quantities have a common factor a, and when stripped of this become $b + d$, and $b - d$, which are prime to each other. Therefore the L.C.M. required is

$$a\,(b + d)\,(b - d) \text{ or } ab^2 - ad^2.$$

The following method is generally applicable to all quantities Simple or Compound.

116. *To find the Least Common Multiple of two quantities, or the least quantity which is divisible by each of them without remainder.*

Let a and b be the two quantities, x their greatest common measure, m their least common multiple, and let m contain a, p times, and b, q times, that is, let $m = pa = qb$; then dividing the two latter equal quantities by pb (Art. 82) $\frac{a}{b} = \frac{q}{p}$; and since m is the least possible, p and q are the least possible; therefore $\frac{q}{p}$ is the fraction $\frac{a}{b}$ in its lowest terms,* and consequently $q = \frac{a}{x}$; hence $m = qb = \frac{a}{x} \times b$.

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* For, if not, let some other fraction $\frac{q'}{p'}$ be the fraction $\frac{a}{b}$ in its lowest terms; then since $\frac{q'}{p'} = \frac{a}{b}$, multiplying these equal quantities by $p'b$ (Art. 81) $q'b = p'a$, or there are common multiples of a and b less than pa and qb, which is impossible, since pa and qb are the *least*.

The rule here proved may be thus enunciated:—

Find the G.C.M. of the two proposed quantities; divide one of them by this G.C.M.; and multiply the quotient thus obtained by the other quantity. The product is the Least Common Multiple required.

Ex. Required the Least Common Multiple of $a^4 - x^4$ and $a^3 - a^2x - ax^2 + x^3$.

The G.C.M. of these two quantities (See Art. 105, Ex. 2), is $a^2 - x^2$; and $(a^4 - x^4) \div (a^2 - x^2) = a^2 + x^2$. Therefore the Least Common Multiple required

$$\begin{aligned} &= (a^2 + x^2).(a^3 - a^2x - ax^2 + x^3) \\ &= a^5 - a^4x - ax^4 + x^5. \end{aligned}$$

117. *Every other common multiple of* a *and* b *is a multiple of* m.

Let n be any other common multiple of the two quantities; and, if possible, let m be contained r times in n, with a remainder s, which is less than m; then $n - rm = s$; and since a and b measure n and rm, they measure $n - rm$, or s (Art. 104); that is, they have a common multiple less than m, which is contrary to the supposition.

118. *To find the Least Common Multiple of three quantities* a, b, c, *find* m *the Least Common Multiple of* a *and* b, *and* n *the Least Common Multiple of* m *and* c; *then* n *is the Least Common Multiple sought.*

For every common multiple of a and b is a multiple of m (Art. 117); therefore every common multiple of a, b, and c is a multiple of m and c; also every multiple of m and c is a multiple of a, b, and c; consequently the Least Common Multiple of m and c is the Least Common Multiple of a, b, and c.

And similarly if there be four or more quantities of which the Least Common Multiple is required.

Ex. Required the Least Com. Mult. of $x^3 - a^2x - ax^2 + a^3$, $x^4 - a^4$, and $ax^3 + a^3x - a^2x^2 - a^4$.

Here $ax^3 + a^3x - a^2x^2 - a^4 = a(x^3 + a^2x - ax^2 - a^3)$;

$\therefore$ to find the G.C.M. of this quantity and the first, reject the factor a;

$$\begin{array}{r|l} x^3 - a^2x - ax^2 + a^3 \,) & x^3 + a^2x - ax^2 - a^3 \,(1 \\ & x^3 - a^2x - ax^2 + a^3 \\ \hline & \quad 2a^2x - 2a^3 = 2a^2(x - a) \end{array}$$

$$x-a)\ x^3-a^2x-ax^2+a^3\ (x^2-a^2$$
$$x^3-ax^2$$
$$-a^2x+a^3$$
$$-a^2x+a^3$$
$$*\quad *$$

$\therefore$ $x-a$ is the G. C. M. of the first and last of the proposed quantities; and their least com. mult. is

$$(ax^3+a^3x-a^2x^2-a^4)(x^2-a^2)\ldots\ldots(1).$$

The other quantity is

$$(x^2+a^2)(x^2-a^2)\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

The G. C. M. of (1) and (2) is $(x^2-a^2)\times$ the G. C. M. of $ax^3+a^3x-a^2x^2-a^4$ and x^2+a^2. Rejecting the factor a in the former quantity,

$$x^2+a^2)\ x^3+a^2x-ax^2-a^3\ (x-a$$
$$x^3+a^2x$$
$$-ax^2-a^3$$
$$-ax^2-a^3$$
$$*\quad *$$

$\therefore$ the G. C. M. of (1) and (2) is $(x^2+a^2)(x^2-a^2)$;

$\therefore$ least com. mult. required is

$$(ax^3+a^3x-a^2x^2-a^4)(x^2-a^2),$$
$$\text{or } ax^5-a^2x^4-a^5x+a^6.$$

119. A more expeditious method of applying the preceding rule to find the Least Com. Mult., when it can readily be done, is that of resolving each quantity into its component factors, as follows:—taking the last Example,

$$(1)\quad x^3-a^2x-ax^2+a^3=x^2(x-a)-a^2(x-a)$$
$$=(x^2-a^2)(x-a).$$
$$(2)\quad x^4-a^4=(x^2+a^2)(x^2-a^2)=(x^2+a^2)(x-a)(x+a).$$
$$(3)\quad ax^3+a^3x-a^2x^2-a^4=ax^2(x-a)+a^3(x-a)$$
$$=a(x^2+a^2)(x-a).$$

Now the G. C. M. of (2) and (3) is $(x^2+a^2)(x-a)$;

$\therefore$ least com. mult. of (2) and (3) is $a(x^4-a^4)\ldots(4)$.

Again, the G. C. M. of (1) and (4) is x^2-a^2;

$\therefore$ least com. mult. required is $a(x^4-a^4)(x-a)$,

$$\text{or } ax^5-a^2x^4-a^5x+a^6.$$

ADDITION AND SUBTRACTION OF FRACTIONS.

120. *If the fractions to be added together have a common denominator, their sum is found by adding the numerators together for a new numerator and retaining the common denominator.*

Thus $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$. This follows from the principle laid down in Art. 99.

121. If the fractions have not a common denominator they must be transformed to others of the same value, which have a common denominator (Art. 112...114), and then the addition may take place as before.

Ex. 1. $$\frac{a}{b} + \frac{c}{d} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}.$$

Ex. 2. $$\frac{1}{a+b} + \frac{1}{a-b} = \frac{a-b}{a^2-b^2} + \frac{a+b}{a^2-b^2}$$
$$= \frac{a-b+a+b}{a^2-b^2}$$
$$= \frac{2a}{a^2-b^2}.$$

Ex. 3. $$a + \frac{e}{f} = \frac{af}{f} + \frac{e}{f} = \frac{af+e}{f}.$$

Here a is considered as a fraction whose denominator is 1.

Ex. 4. $$2 + \frac{a+b}{a-b} + \frac{a-b}{a+b} = \frac{2a^2-2b^2}{a^2-b^2} + \frac{a^2+2ab+b^2}{a^2-b^2} + \frac{a^2-2ab+b^2}{a^2-b^2} = \frac{2a^2-2b^2+a^2+2ab+b^2+a^2-2ab+b^2}{a^2-b^2}$$
$$= \frac{4a^2}{a^2-b^2}.$$

Ex. 5. $$\frac{a}{mx} + \frac{b}{my} + \frac{c}{mz} = \frac{ayz}{mxyz} + \frac{bxz}{mxyz} + \frac{cxy}{mxyz}$$
$$= \frac{ayz+bxz+cxy}{mxyz}.$$

Ex. 6. Required the sum of $\frac{2}{x^3+x^2+x+1}$ and $\frac{3}{x^3-x^2+x-1}$.

By Art. 116 or Art. 119 the Least Com. Mult. of the denominators is found to be x^4-1; therefore the sum required is

$$\frac{2(x-1)}{x^4-1}+\frac{3(x+1)}{x^4-1}$$
$$=\frac{2x-2+3x+3}{x^4-1}$$
$$=\frac{5x+1}{x^4-1}.$$

122. ***If two fractions have a common denominator, their difference is found by taking the difference of the numerators for a new numerator and retaining the common denominator.***

Thus $\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$. (See Art. 99.)

123. If they have not a common denominator, they must be transformed to others of the same value, which have a common denominator, and then the subtraction may take place as before.

Ex. 1. $\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}$.

Ex. 2. $a-\frac{cd}{b}=\frac{ab}{b}-\frac{cd}{b}=\frac{ab-cd}{b}$.

Ex. 3. $\frac{a}{b}-\frac{c+d}{c-d}=\frac{ac-ad}{bc-bd}-\frac{bc+bd}{bc-bd}=\frac{ac-ad-bc-bd}{bc-bd}$.

The sign of bd in the numerator is negative, because every part of the latter fraction is to be taken from the former. (See Art. 87).

Ex. 4. $\frac{a+b}{a-b}-\frac{a-b}{a+b}=\frac{a^2+2ab+b^2}{a^2-b^2}-\frac{a^2-2ab+b^2}{a^2-b^2}$

$$=\frac{a^2+2ab+b^2-a^2+2ab-b^2}{a^2-b^2}=\frac{4ab}{a^2-b^2}.$$

Ex. 5. $\frac{3+2x}{1-x^2}-\frac{2}{1-x}=\frac{3+2x}{1-x^2}-\frac{2(1+x)}{1-x^2}$

$$=\frac{3+2x-2-2x}{1-x^2}$$
$$=\frac{1}{1-x^2}.$$

MULTIPLICATION AND DIVISION OF FRACTIONS.

124. *To multiply a fraction by any quantity multiply the numerator by that quantity and retain the denominator.*

Thus $\frac{a}{b} \times c = \frac{ac}{b}$. For if the quantity to be divided be c times as great as before, and the divisor the same, the quotient must be c times as great.

125. Cor. 1. $\frac{a}{b} \times b = \frac{ab}{b} = a$; that is, if a fraction be multiplied by its denominator, the product is the numerator.

126. Cor. 2. The result is the same, whether the numerator be multiplied by a given quantity, or the denominator divided by it.

Let the fraction be $\frac{ad}{bc}$, and let its numerator be multiplied by c, the result is $\frac{adc}{bc}$, or $\frac{ad}{b}$ (Art. 101), the quantity which arises from the division of its denominator by c.

127. *The product of two fractions is found by multiplying the numerators together for a new numerator, and the denominators for a new denominator.*

Let $\frac{a}{b}$ and $\frac{c}{d}$ be the two fractions; then $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$. For if $x = \frac{a}{b}$, and $y = \frac{c}{d}$, by multiplying the equal quantities, x and $\frac{a}{b}$, by b, $bx = a$ (Art. 81); in the same manner, $dy = c$; therefore, by the same axiom, $bx \, . \, dy = ac$, that is, $bdxy = ac$; dividing these equal quantities, $bdxy$ and ac, by bd (Art. 82), we have $xy = \frac{ac}{bd}$,

$$\text{or } \frac{a}{b} \, . \, \frac{c}{d} = \frac{ac}{bd}.$$

128. *To divide a fraction by any quantity multiply the denominator by that quantity, and retain the numerator.*

The fraction $\frac{a}{b}$ divided by c is $\frac{a}{bc}$. Because $\frac{a}{b} = \frac{ac}{bc}$, and a

c^{th} part of this is $\dfrac{a}{bc}$; the quantity to be divided being a c^{th} part of what it was before, and the divisor the same.

129. Cor. The result is the same, whether the denominator is multiplied by the quantity, or the numerator divided by it.

Let the fraction be $\dfrac{ac}{bd}$; if the denominator be multiplied by c, it becomes $\dfrac{ac}{bdc}$ or $\dfrac{a}{bd}$, the quantity which arises from the division of the numerator by c.

130. *To divide one fraction by another invert the numerator and denominator of the divisor, and use it as a multiplier according to the rule for multiplication.*

Let $\dfrac{a}{b}$ and $\dfrac{c}{d}$ be the two fractions, then $\dfrac{a}{b}\div\dfrac{c}{d}=\dfrac{a}{b}\times\dfrac{d}{c}=\dfrac{ad}{bc}$.

For if $x=\dfrac{a}{b}$, and $y=\dfrac{c}{d}$, then as in Art. 127, $bx=a$, and $dy=c$; also $bdx=ad$, and $bdy=bc$; therefore by Art. 82, $\dfrac{bdx}{bdy}=\dfrac{ad}{bc}$,

or $\dfrac{x}{y}=\dfrac{ad}{bc}$, or $\dfrac{a}{b}\div\dfrac{c}{d}=\dfrac{ad}{bc}$.

131. *To prove the Rules for Multiplication and Division of Decimal Fractions.**

By the Definition of a Decimal Fraction (Art. 39,) $\dfrac{P}{10^m}$ is equivalent to a Decimal having m decimal places, and P being the number which the decimal represents excluding the decimal point; similarly $\dfrac{Q}{10^n}$ is equivalent to a decimal which has n decimal places.

Now $\dfrac{P}{10^m}\times\dfrac{Q}{10^n}=\dfrac{PQ}{10^{m+n}}=$ a decimal fraction formed by multiplying P into Q and then marking off $\overline{m+n}$ decimal places:—which proves the rule for multiplication.

Again,
$$\begin{aligned}\frac{P}{10^m}\div\frac{Q}{10^n}&=\frac{P}{Q}\times\frac{1}{10^{m-n}}, &&\text{if } m>n\\ &=\frac{P}{Q}, &&\text{if } m=n\\ &=\frac{P}{Q}\cdot 10^{n-m}, &&\text{if } m<n.\end{aligned}$$

Hence the rule for division is, Divide P by Q, as in whole numbers; id then,

If $m > n$, or the number of decimal places in the dividend > the imber in the divisor, mark off $\overline{m-n}$ decimal places in the quotient.

If $m = n$, the quotient is an integer.

If $m < n$, affix $\overline{n-m}$ ciphers to the right of the quotient.

132. The rule for multiplying the powers of the same quanty (Art. 91), will hold when one or both of the indices are negative.

Thus $a^m \times a^{-n} = a^{m-n}$; for $a^m \times a^{-n} = a^m \times \frac{1}{a^n}$ (Art. 65.) $= \frac{a^m}{a^n} = a^{m-n}$ Art. 96.). In the same manner $x^3 \times x^{-5} = \frac{x^3}{x^5} = \frac{1}{x^2} = x^{-2}$.

Again, $a^{-m} \times a^{-n} = a^{-(m+n)}$; because

$$a^{-m} \times a^{-n} = \frac{1}{a^m} \times \frac{1}{a^n} \text{ (Art. 65.) } = \frac{1}{a^{m+n}} = a^{-(m+n)}.$$

133. Cor. If $m = n$, $a^m \times a^{-m} = a^{m-m} = a^0$; also $a^m \times a^{-m} = \frac{a^m}{a^m}$ 1; therefore $a^0 = 1$, according to the notation adopted in Arts. 3, 65.

134. The rule for dividing any power of a quantity by any her power of the same quantity (Art. 96.) holds, whether those owers are positive or negative.

Thus $a^m \div a^{-n} = a^m \div \frac{1}{a^n}$ (Art. 65), $= a^m \times a^n = a^{m+n}$.

Again, $a^{-m} \div a^{-n} = \frac{1}{a^m} \div \frac{1}{a^n} = \frac{a^n}{a^m}$ (Art. 130.) $= a^{n-m}$.

135. Cor. Hence it appears, that a *factor* may be transferred om the numerator of a fraction to the denominator, and *vice rsâ*, by changing the sign of its index.

Thus $\frac{a^m \,.\, a^n}{b^p} = \frac{a^m}{b^p a^{-n}}$; and $\frac{a^m}{a^n b^p} = \frac{a^m \,.\, a^{-n}}{b^p} = a^m \,.\, a^{-n} \,.\, b^{-p}$.

Also $\frac{4ab^{-2}cx^{-3}}{5a^{-2}y^{-1}d^2} = \frac{4a \,.\, a^2cy}{5b^2d^2x^3} = \frac{4a^3cy}{5b^2d^2x^3}$.

INVOLUTION AND EVOLUTION.

136. If a quantity be continually multiplied by itself, it is said to be involved, or raised; and the power to which it is raised is expressed by the number of times the quantity has been employed in the multiplication.

Thus $a \times a$, or a^2, is called the second power of a; $a \times a \times a$, or a^3, the third power; $a \times a$ ... to n factors, or a^n, the n^{th} power of a.

137. If the quantity to be involved be negative, the signs of the even powers will be positive, and the signs of the odd powers negative.

For $-a \times -a = a^2$, $-a \times -a \times -a = -a^3$, &c.

138. A *simple* quantity is raised to any power by multiplying the index of every factor in the quantity by the exponent of that power, and prefixing the proper sign determined by the last article.

Thus a^m raised to the n^{th} power is a^{mn}; because $a^m \times a^m \times a^m \ldots$ to n factors, by the rule of multiplication, is $a^{m+m+\&c.\text{ to } n \text{ terms}}$, or a^{mn}. Also $(ab)^n = ab \times ab \times ab \times$ &c. to n factors, or $a \times a \times a$ ..... to n factors $\times b \times b \times b \ldots$ to n factors (Art. 88) $= a^n \times b^n$. And $a^2 b^3 c$ raised to the fifth power is $a^{10} b^{15} c^5$. Also $-a^m$ raised to the n^{th} power is $\pm a^{mn}$; where the positive or negative sign is to be prefixed, according as n is an even or odd number.

This last rule is algebraically expressed thus,

$$(-a^m)^n = (-1)^n . a^{mn};$$

the coefficient $(-1)^n$ being sufficient to express that the positive or negative sign is to be prefixed to a^{mn}, according as n is an even or an odd number.

139. If the quantity to be involved be a *fraction*, both the numerator and denominator must be raised to the proposed power (Art. 127).

$$\text{For } \left(\frac{a}{b}\right)^n = \frac{a}{b}.\frac{a}{b}.\frac{a}{b} \ldots \text{ to } n \text{ factors} = \frac{a.a.a \ldots \text{ to } n \text{ factors}}{b\ b.b \ldots \text{ to } n \text{ factors}} = \frac{a^n}{b^n}.$$

140. If the quantity proposed be a *compound* one, the involution may either be represented by the proper index, or it may actually take place.

1. Let $a+b$ be the quantity to be raised to any power.

$$\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline \end{array}$$

$(a+b)^2$ or $a^2+2ab+b^2$ the square, or 2^d power.

$$\begin{array}{l} a+b \\ \hline a^3+2a^2b+ab^2 \\ \quad +a^2b+2ab^2+b^3 \\ \hline \end{array}$$

$(a+b)^3$ or $a^3+3a^2b+3ab^2+b^3$ the 3^d power.

$$\begin{array}{l} a+b \\ \hline a^4+3a^3b+3a^2b^2+ab^3 \\ \quad +a^3b+3a^2b^2+3ab^3+b^4 \\ \hline \end{array}$$

$(a+b)^4$ or $\underline{a^4+4a^3b+6a^2b^2+4ab^3+b^4}$ the 4th power; and so on for any higher power.

If b be negative, or the quantity to be involved be $a-b$, whereever an odd power of b enters, the sign of the term will be negative (Art. 137).

Hence $(a-b)^2=a^2-2ab+b^2$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3$

$(a-b)^4=a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

2. Let $a+b+c$ be the quantity to be raised to any power.

$$\begin{array}{l} a+b+c \\ a+b+c \\ \hline a^2+ab+ac \\ \quad +ab+b^2+bc \\ \qquad +ac+bc+c^2 \\ \hline \end{array}$$

$(a+b+c)^2$ or $a^2+b^2+c^2+2ab+2bc+2ac$ the square, or 2^d power.

$$\begin{array}{l} a+b+c \\ \hline a^3+ab^2+ac^2+2a^2b+2abc+2a^2c \\ +a^2b+b^3+bc^2+2ab^2+2b^2c+2abc \\ +a^2c+b^2c+c^3+2abc+2bc^2+2ac^2 \\ \hline \end{array}$$

$(a+b+c)^3$ or $a^3+b^3+c^3+3(a^2b+a^2c+ab^2+ac^2+b^2c+bc^2)+6abc$ the cube or 3^{rd} power; and so on for any higher power.

141. By using *brackets* the involution of a quantity consisting of more than two terms may be always made to depend upon that of a binomial, and thereby the operation be much abridged.

$$\begin{aligned}\text{Thus } (a+b+c)^2 &= (a+\overline{b+c})^2\\ &= a^2+(b+c)^2+2a(b+c)\\ &= a^2+b^2+c^2+2bc+2ab+2ac.\end{aligned}$$

$$\begin{aligned}(a+b+c)^3 &= (\overline{a+b}+c)^3\\ &= (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3\\ &= a^3+3a^2b+3ab^2+b^3+3(a^2+b^2+2ab)c\\ &\qquad\qquad +3ac^2+3bc^2+c^3\\ &= a^3+b^3+c^3+3(a^2b+ab^2+a^2c+ac^2+b^2c+bc^2)+6abc.\end{aligned}$$

$$\begin{aligned}\text{Again } (a+b+c+d)^2 &= (\overline{a+b}+\overline{c+d})^2\\ &= (a+b)^2+(c+d)^2+2(a+b)(c+d)\\ &= a^2+b^2+2ab+c^2+d^2+2cd+2ac+2ad+2bc+2bd\\ &= a^2+b^2+c^2+d^2+2(ab+ac+ad+bc+bd+cd);\end{aligned}$$

and so on.

142. Since $(a+b+c+d+\&c.)^2$ may be thus arranged,

$$a^2+2a(b+c+d+\&c.)+b^2+2b(c+d+\&c.)+c^2+2c(d+\&c.)+\&c.$$

the square of any multinomial may be readily found by the following RULE:

Square each term, and multiply twice that term into the sum of the several terms that come after; the sum of all the results so obtained will be the square of the whole quantity. Thus,

$$\begin{aligned}\text{Ex. 1.}\quad (a+b+c+d)^2 &= a^2+2a(b+c+d)\\ &\quad +b^2+2b(c+d)\\ &\quad +c^2+2cd\\ &\quad +d^2.\end{aligned}$$

$$\begin{aligned}\text{Ex. 2.}\quad (a-b-c+d)^2 &= a^2+2a(d-c-b)\\ &\quad +b^2-2b(d-c)\\ &\quad +c^2-2cd\\ &\quad +d^2.\end{aligned}$$

$$\begin{aligned}\text{Ex. 3.}\quad (1+\tfrac{1}{2}x+\tfrac{1}{3}x^2)^2 &= 1+x+\frac{2}{3}x^2\\ &\qquad +\frac{1}{4}x^2+\frac{1}{3}x^3\\ &\qquad\qquad +\frac{1}{9}x^4\\ &= 1+x+\frac{11}{12}x^2+\frac{1}{3}x^3+\frac{1}{9}x^4.\end{aligned}$$

Ex. 4. $(1+x+2x^2+3x^3+4x^4+5x^5+\ldots)^2$

$$\begin{aligned}&=1+2(x+2x^2+3x^3+4x^4+5x^5+6x^6+\ldots)\\&+x^2+2x(2x^2+3x^3+4x^4+5x^5+\ldots)\\&+4x^4+4x^2(3x^3+4x^4+\ldots)\\&+9x^6+6x^3(4x^4+\ldots)\\&+\ldots\ldots\ldots\ldots\ldots\ldots\\&=1+2x+5x^2+10x^3+18x^4+30x^5+47x^6+\ldots\end{aligned}$$

N.B. It will be found useful *to commit to memory* the following sults:—

$$(A+B)^3=A^3+B^3+3AB(A+B)\ldots\ldots(1),$$

$$(A-B)^3=A^3-B^3-3AB(A-B)\ldots\ldots(2),$$

hatever quantities A and B may represent, *simple* or *compound*.

Thus, Ex. 1. Let the cube of $1+x+x^2$ be required.

$$\begin{aligned}(1+x+x^2)^3&=(\overline{1+x}+x^2)^3\\&=(1+x)^3+(x^2)^3+3(1+x)x^2(1+x+x^2),\text{ by (1)}\\&=1+x^3+3x+3x^2+x^6+(3x^2+3x^3)(1+x+x^2)\\&=1+x^3+3x+3x^2+x^6+3x^2+3x^3+3x^4+3x^3+3x^4+3x^5\\&=1+3x+6x^2+7x^3+6x^4+3x^5+x^6.\end{aligned}$$

Ex. 2. Required the cube of $\sqrt[3]{a+x}-\sqrt[3]{a-x}$

$$\overline{a+x}-\sqrt[3]{a-x})^3=a+x-\overline{a-x}-3\sqrt[3]{a+x}\,.\,\sqrt[3]{a-x}\times(\sqrt[3]{a+x}-\sqrt[3]{a-x}),\text{ by (2)}$$

$$=2x-3\,.\,\sqrt[3]{a+x}\,.\,\sqrt[3]{a-x}\,.\,\{\sqrt[3]{a+x}-\sqrt[3]{a-x}\}.$$

143. EVOLUTION, or the extraction of roots, is the method determining a quantity which raised to a proposed power will oduce a given quantity.

144. Since the n^{th} power of a^m is a^{mn} (Art. 138), the n^{th} root a^{mn} must be a^m; that is, to extract any root of a single quantity, must divide the index of that quantity by the index of the root quired. Thus $\sqrt[3]{a^6}=a^2$, $\sqrt[4]{a^{12}}=a^3$; &c.

145. When the index of the quantity is not exactly divisible the number which expresses the root to be extracted, that root ust be represented according to the notation pointed out in rt. 70. Thus the square, cube, fourth, n^{th}, root of a^2+x^2, are spectively represented by

$$(a^2+x^2)^{\frac{1}{2}},\quad(a^2+x^2)^{\frac{1}{3}},\quad(a^2+x^2)^{\frac{1}{4}},\quad(a^2+x^2)^{\frac{1}{n}};$$

the same roots of $\frac{1}{a^2+x^2}$, or $(a^2+x^2)^{-1}$, are represented by

$$(a^2+x^2)^{-\frac{1}{2}},\ (a^2+x^2)^{-\frac{1}{3}},\ (a^2+x^2)^{-\frac{1}{4}},\ (a^2+x^2)^{-\frac{1}{n}}.$$

146. If the root to be extracted be expressed by an odd number, the sign of the root will be the same with the sign of the proposed quantity, as appears by Art. 137. Thus $\sqrt[3]{8}$ is 2; $\sqrt[3]{-8}$ is -2; &c.

147. If the root to be extracted be expressed by an even number, and the quantity proposed be positive, the root may be either positive or negative. Because either a positive or negative quantity raised to such a power is positive (Art. 137). Thus $\sqrt{a^2}$ is $\pm a$; $\sqrt[4]{(a+x)^8}$ is $\pm(a+x)^2$; &c.

148. If the root proposed to be extracted be expressed by an even number, and the sign of the proposed quantity be negative, the root cannot be extracted; because no quantity raised to an even power can produce a negative result. Such roots are called *impossible*.

149. Any root of a *product* may be found by taking that root of each factor, and multiplying the roots, so taken, together.

For $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}}$; because each of these quantities, raised to the n^{th} power, is ab (Art. 138).

Thus $\sqrt[2]{a^2b^4} = \sqrt[2]{a^2} . \sqrt[2]{b^4} = ab^2$; and $\sqrt[5]{a^{10}b^{15}c^5} = a^2b^3c$.

COR. If $a = b$, then $a^{\frac{1}{n}} \times a^{\frac{1}{n}} = a^{\frac{2}{n}}$; and in the same manner $a^{\frac{r}{n}} \times a^{\frac{s}{n}} = a^{\frac{r+s}{n}}$.

150. To obtain a clear and comprehensive view of the *Theory of Fractional and Negative Indices*, let the reader revert to Arts. 63, 65, and 70. The first Article is strictly so called a Definition; but the others are not; at least they are not *arbitrary* definitions, but are dependent upon something which does not appear. Thus it is strictly a Definition, that

$a . a . a$ shall be represented by a^3,

and $a . a . a$...to n factors.........by a^n, if n be any positive integer.

Hence it is easily shewn that, when m and n are positive integers,

$$a^m \, . \, a^n = a^{m+n} \ldots\ldots (1),$$

$$\text{and } \frac{a^m}{a^n} = a^{m-n} \ldots\ldots (2), \text{ (Arts. 91, 96).}$$

Now, the assumptions in Arts. 65 and 70 are made subject to these two Rules, that is, made so that these two rules may hold true for fractional and negative indices, as well as for positive integers. Thus, if Rule (1) be true for negative integers as indices,

$$a^m \, . \, a^{-n} = a^{m-n};$$

$$\text{and if } a^{-n} \text{ stand for } \frac{1}{a^n},$$

$$a^m \, . \, a^{-n} = a^m \, . \, \frac{1}{a^n} = \frac{a^m}{a^n} = a^{m-n}, \text{ as before, by (2).}$$

$$\text{Again, if } a^{\frac{1}{2}} \text{ stand for } \sqrt{a},$$

$$\text{then } a^{\frac{1}{2}} \, . \, a^{\frac{1}{2}} = \sqrt{a} \, . \, \sqrt{a},$$

$$\text{But } \sqrt{a} \cdot \sqrt{a} = a, \text{ by definition,}$$

and $a^{\frac{1}{2}} \, . \, a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a$, if Rule (1) be true for fractional indices.

$$\text{Similarly if } a^{\frac{1}{3}} \text{ stand for } \sqrt[3]{a},$$

$$a^{\frac{1}{3}} \, . \, a^{\frac{1}{3}} \, . \, a^{\frac{1}{3}} = \sqrt[3]{a} \, . \sqrt[3]{a} \, . \sqrt[3]{a}.$$

$$\text{But } \sqrt[3]{a} \, . \sqrt[3]{a} \, . \sqrt[3]{a} = a, \text{ by definition,}$$

and $a^{\frac{1}{3}} \, . \, a^{\frac{1}{3}} \, . \, a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a$, if Rule (1) be true for fractional indices.

And generally,

$$\text{since } \sqrt[n]{a} \, . \sqrt[n]{a} \, . \sqrt[n]{a} \, . \text{ \&c. to } n \text{ factors} = a;$$

Also $a^{\frac{1}{n}} \, . \, a^{\frac{1}{n}} \, . \, a^{\frac{1}{n}} \, .$ &c. to n factors $= a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots \text{to } n \text{ terms}} = a^{\frac{n}{n}} = a$, if Rule (1) be true for fractional indices,

$$\therefore \; a^{\frac{1}{n}} \text{ may stand for } \sqrt[n]{a}.$$

Again, since $a^{\frac{m}{n}} = a^{\frac{1}{n}+\frac{1}{n}+\ldots \text{to } m \text{ terms}} = a^{\frac{1}{n}} \, . \, a^{\frac{1}{n}} \, . \, a^{\frac{1}{n}} \ldots$ to m factors;

$$\therefore \; a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m \ldots\ldots\ldots (4).$$

$$\text{And again, } a^{\frac{m}{n}} \, . \, a^{\frac{p}{q}} = a^{\frac{mq}{nq}} \, . \, a^{\frac{np}{nq}}$$

$$= \left(a^{\frac{1}{nq}}\right)^{mq} . \left(a^{\frac{1}{nq}}\right)^{np}, \text{ by (4),}$$

$$= \left(a^{\frac{1}{nq}}\right)^{mq+np}, \text{ by (1)},$$
$$= a^{\frac{mq+np}{nq}}, \text{ by (4)}$$
$$= a^{\frac{m}{n}+\frac{p}{q}}.$$

Hence it appears that the Definitions respecting *negative* and *fractional* Indices are not made arbitrarily, but have reference to the RULES already determined for the multiplication and division of powers, when the Indices are *positive integers*.

Therefore $a^m . a^n = a^{m+n}$, whether m and n be whole or fractional, positive or negative.

And similarly $a^m \div a^n = a^{m-n}$, whether m and n be whole or fractional, positive or negative.

151. Any root of a *fraction* may be found by taking that root of both the numerator and denominator (Art. 139.), that is, the root of the numerator for a new numerator, and the same root of the denominator for a new denominator.

Thus the cube root of $\frac{a^2}{b^2}$ is $\frac{a^{\frac{2}{3}}}{b^{\frac{2}{3}}}$, or $a^{\frac{2}{3}} \times b^{-\frac{2}{3}}$; and $\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$, or $a^{\frac{1}{n}} \times b^{-\frac{1}{n}}$.

152. *To extract the square root of a* compound *quantity.*

Since the square root of $a^2 + 2ab + b^2$ is $a + b$ (Art. 140), whatever be the values of a and b, we may obtain a general rule for the extraction of the square root, by observing in what manner a and b may be derived from $a^2 + 2ab + b^2$.

$$\begin{array}{r|l} & a^2 + 2ab + b^2 \,(a + b \\ & a^2 \\ \hline 2a + b) & 2ab + b^2 \\ & 2ab + b^2 \\ \hline & \;\;*\;\;\;\;\; * \end{array}$$

Having arranged the terms according to the dimensions of one letter, a, the square root of the first term, a^2, is a, the first term in the root; subtract its square from the whole quantity, and bring down the remainder $2ab + b^2$; divide $2ab$ by $2a$, and the result is b, the other term in the root; then multiply the sum of twice the first term and the second, $2a + b$, by the second, b, and subtract this product, $2ab + b^2$, from the remainder. If there be more terms, consider $a + b$ as a new value of a; and its square, that is, $a^2 + 2ab + b^2$, having by the first part of the process been subtracted from the proposed quantity, divide the remainder by the double of this new value of a, for a new term in the root; and for a new subtrahend multiply this term by twice the sum of the former terms increased by this term. The process must be repeated till the root, or the necessary approximation to the root, is obtained*.

Ex. 1. To extract the square root of $a^2+2ab+b^2+2ac+2bc+c^2$.

Having arranged the terms of the proposed quantity according to the dimensions of one letter, a, it becomes

$$a^2+2(b+c)\,a+b^2+2bc+c^2.$$

Hence, following the rule, we have

$$\begin{array}{r|l}
 & a^2+2(b+c)\,a+b^2+2bc+c^2\ \big(\,a+(b+c) \\
 & a^2 \\
\hline
2a+(b+c)\,\big) & 2(b+c)\,a+b^2+2bc+c^2 \\
 & 2(b+c)\,a+b^2+2bc+c^2 \\
\hline
 & \quad * \quad * \quad *
\end{array}$$

$\therefore\ a+b+c$ is the root required.

Ex. 2. To extract the square root of $a^2-ax+\frac{x^2}{4}$.

$$\begin{array}{r|l}
 & a^2-ax+\frac{x^2}{4}\ \big(\,a-\frac{x}{2},\ \text{the root required.} \\
 & a^2 \\
\hline
2a-\frac{x}{2}\,\big) & -ax+\frac{x^2}{4} \\
 & -ax+\frac{x^2}{4} \\
\hline
 & \quad * \quad *
\end{array}$$

---

* That the Rule may be thus extended will be obvious from comparing the form of the squares of $a+b+c$, $a+b+c+d$, &c. with that of $a+b$, from which the Rule was deduced. For

Ex. 3. To extract the square root of $1+x$.

$$
\begin{array}{rl}
 & 1+x\,\Big(\,1+\frac{x}{2}-\frac{x^2}{8}+\&c. \\
 & 1 \\
\cline{2-2}
2+\frac{x}{2}\Big) & \quad x \\
 & \quad x+\frac{x^2}{4} \\
\cline{2-2}
2+x-\frac{x^2}{8}\Big) & \quad -\frac{x^2}{4} \\
 & \quad -\frac{x^2}{4}-\frac{x^3}{8}+\frac{x^4}{64} \\
\cline{2-2}
 & \qquad \frac{x^3}{8}-\frac{x^4}{64}\ \&c.
\end{array}
$$

Since $1+\frac{x}{2}=\sqrt{1+x+\frac{x^2}{4}}$; $1+\frac{x}{2}-\frac{x^2}{8}=\sqrt{1+x-\frac{x^3}{8}+\frac{x^4}{64}}$; and so on; if x be of such a magnitude that each of these successive quantities under the root differs from $1+x$ less than the preceding one, the continued series of terms $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-$&c. will be an *approximation* to the true root of $1+x$; that is, the more terms of the series are taken, the less will their sum differ from the square root of $1+x$. And, since it is evident that no expression whatsoever, simple or compound, multiplied into itself, can produce $1+x$, therefore an *approximation* to the root is all that can be required.

But if x be *not* of such a magnitude as is here supposed, then it is clear that the operation performed upon $1+x$ leads to a result which is not even an *approximation* to its square root, because as more terms of that result are taken and squared we obtain a quantity which recedes farther and farther from $1+x$. Its proper interpretation, in this stage of the subject, cannot well be given, and must be deferred.

$$
\begin{aligned}
(a+b+c)^2 &= (a+b)^2+2(a+b)c+c^2, \text{ (Art. 141.)}\\
&= a^2+(2a+b)b+\{2(a+b)+c\}c.\\
(a+b+c+d)^2 &= (a+b+c)^2+2(a+b+c)d+d^2\\
&= a^2+(2a+b)b+\{2(a+b)+c\}c+\{2(a+b+c)+d\}d;
\end{aligned}
$$

and so on; from which method of exhibiting the square of a *multinomial* the rule for extracting the square root is evidently seen to hold whatever be the number of terms in the root.

Cor. If the approximate square root of *any* binomial, as $a+b$, be required; since $a+b=a\left(1+\frac{b}{a}\right)$; and therefore $\sqrt{a+b}=\sqrt{a}\times\sqrt{1+\frac{b}{a}}$ (Art. 149.); we have by the last Example, putting $\frac{b}{a}$ in the place of x,

$$\sqrt{1+\frac{b}{a}}=1+\frac{b}{2a}-\frac{b^2}{8a^2}+\&c.$$

$$\therefore\ \sqrt{a+b}=a^{\frac{1}{2}}+\frac{b}{2a^{\frac{1}{2}}}-\frac{b^2}{8a^{\frac{3}{2}}}+\&c.$$

153. It appears from Ex. 2, that a trinomial, $a^2-ax+\frac{x^2}{4}$, in which four times the product of the first and last terms is equal to the square of the middle term, is a complete square.

Thus $4x^2+4cx+c^2$ is a complete square, viz. $(2x+c)^2$ because $4\times 4x^2\times c^2=16c^2x^2=(4cx)^2$.

Similarly $x^2-px+\frac{p^2}{4}$ is a complete square, viz. $\left(x-\frac{p}{2}\right)^2$.

154. The method of extracting the cube root is discovered in the same manner as that for the square root.

The cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$ (Arts. 140, 143); and to obtain $a+b$ from this compound quantity, arrange the terms as before, and the cube root of the first term, a^3, is a, the first term in the root; subtract its cube from the whole quantity, and divide the first term of the remainder by $3a^2$, the result is b, the second term in the root; then subtract $3a^2b+3ab^2+b^3$ from the remainder, and the whole cube of $a+b$ has been subtracted. If any quantity be left, proceed with $a+b$ as a new a, and divide the last remainder by $3(a+b)^2$ for a third term in the root; and thus any number of terms may be obtained*.

$$\begin{array}{r|l}
 & a^3+3a^2b+3ab^2+b^3\,(a+b \\
 & a^3 \\
\hline
3a^2) & 3a^2b+3ab^2+b^3 \\
 & 3a^2b+3ab^2+b^3 \\
\hline
 & *
\end{array}$$

---

* That the rule may be thus extended will be obvious from comparing the form of the cubes of $a+b+c$, $a+b+c+d$, &c., with that of $a+b$ from which the Rule was deduced; For

$$(a+b+c)^3=(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$$
$$=a^3+(3a^2+3ab+b^2)b+\{3(a+b)^2+3(a+b)c+c^2\}c.$$

Similarly $(a+b+c+d)^3=a^3+(3a^2+3ab+b^2)b+\{3(a+b)^2+3(a+b)c+c^2\}c$
$$+\{3(a+b+c)^2+3(a+b+c)d+d^2\}d;$$
and so on.

Ex. To extract the cube root of $8x^3 + 6xy^2 - 12x^2y - y^3$.

$$\begin{array}{rl}
 & 8x^3 - 12yx^2 + 6y^2x - y^3 \,(2x - y \\
a^3 = 8x^3 & \\
\hline
3a^2 = 12x^2) & -12yx^2 + 6y^2x - y^3 \\
\hline
 & -12yx^2 \qquad\qquad = 3a^2b \\
 & \qquad +6y^2x \quad\; = 3ab^2 \\
 & \qquad\qquad -y^3 = b^3 \\
\hline
 & * \quad * \quad * \\
\hline
\end{array}$$

$\therefore\ 2x - y$ is the cube root required.

SCHOLIUM.

155. The rules above laid down for the extraction of the roots of compound quantities are but little used in algebraical operations; but it was necessary to give them at full length, for the purpose of investigating rules for the extraction of the square and cube roots in numbers.

The square root of 100 is 10, of 10000 is 100, of 1000000 is 1000, &c. from which consideration it follows, that the square root of a number less than 100 must consist of only one figure, of a number between 100 and 10000 of two places of figures, of any number from 10000 to 1000000, of three places of figures, &c. If then a point be made over every second figure in any number, beginning with the units, the number of points will shew the number of figures, or places, in the square root. Thus the square root of $4\dot{3}5\dot{7}$ consists of two figures, the square root of $\dot{5}6\dot{4}7\dot{8}$, of three figures, &c.

Ex. 1. Let the square root of 4356 be required.

$$\begin{array}{rl}
 & 4\dot{3}5\dot{6}\,(60 + 6 \text{ or } 66 \\
 & 3600 \\
\hline
120 + 6) & 756 \\
\text{or } 126] & 756 \\
\hline
 & * \\
\end{array}$$

Having pointed it according to the direction, it appears that the root consists of two places of figures; let $a + b$ be the root, where a is the value of the figure in the tens' place, and b of that in the units'; then is a the nearest square root of 4300, which does not exceed the true root; this appears to be 60; subtract the square of 60, (a^2), from the given number, and the remainder is 756; divide this remainder by 120, ($2a$), and the quotient is 6, (the value of b), and the subtrahend, or quantity to be taken from the last remainder 756, is 126×6, $(2a + b)\,b$, or 756.

Hence 66 is the root required.

It is said that a must be the greatest number whose square does not exceed 4300: it evidently cannot be a greater number than this; and if possible let it be some quantity, x, less than this; then since x is in the tens' place and b in the units', $x + b$ is less than a; therefore the square of $x + b$, whatever be the value of b, must be less than a^2, and consequently $x + b$ less than the true root.

If the root consist of three places of figures, let a represent the hundreds, and b the tens; then having obtained a and b as before, let the new value of a be the hundreds and tens together, and find a new value of b for the units: and thus the process may be continued when there are more places of figures in the root.

156. The cyphers being omitted for the sake of expedition, the following rule is obtained from the foregoing process.

Point every second figure beginning with the units' place, dividing by this process the whole number into several periods; find the greatest number whose square is contained in the first period, this is the first figure in the root; subtract its square from the first period, and to the remainder bring down the next period; divide this quantity, omitting the last figure, by twice the part of the root already obtained, and annex the result to the root and also to the divisor; then multiply the divisor, as it now stands, by the part of the root last obtained, for the subtrahend. If there be more periods to be brought down, the operation must be repeated.

$$\begin{array}{r|l} & \dot{4}3\dot{5}6\ (66 \\ & 36 \\ \hline 126) & 756 \\ & 756 \\ \hline & * \end{array}$$

Ex. 2. Let the square root of 611525 be required.

$$\begin{array}{rl}
 & 6\dot{1}1\dot{5}2\dot{5}\ (782 \\
 & 49 \\
\cline{2-2}
148) & 1215 \\
 & 1184 \\
1562) & 3125 \\
 & 3124 \\
\cline{2-2}
 & \qquad 1 \text{ remainder.}
\end{array}$$

The remainder in this example shews that we have not obtained the number which is the *exact* square root of the proposed quantity; but 782 is a near approximation to the square root, being in fact the square root of 611524; and 783 is too great, being the square root of 613089.

157. In extracting the square root of a *decimal*, the pointing must be made the contrary way, beginning with the second place of decimals, and the integral part must be pointed as before, beginning with the units' place: or, if the rule be applied as in whole numbers, care must be taken to have an even number of decimal places, by annexing cyphers to the right (Art. 41); because, if the root have 1, 2, 3, 4, &c. decimal places, the square must have 2, 4, 6, 8, &c. places (Art. 46).

Ex. 3. To extract the square root of 64·853.

$$\begin{array}{rl}
 & 6\dot{4}{\cdot}8\dot{5}3\dot{0}\ (8{\cdot}053\ \&c. \\
 & 64 \\
\cline{2-2}
1605) & 8530 \\
 & 8025 \\
\cline{2-2}
16103) & 50500 \\
 & 48309 \\
\cline{2-2}
 & 2191
\end{array}$$

The remainder in this example appears to be great; but if the decimal point were retained throughout the operation, it would easily be seen that its real value is very small, and that it becomes smaller for every figure that is added to the root.

For every pair of cyphers which we suppose annexed to the decimal another figure is obtained in the root.

And in this and similar cases, when cyphers are added, the root can never terminate, because no figure multiplied by itself can produce a cypher in the units' place.

158. The cube root of 1000 is 10, of 1000000 is 100, &c. therefore the cube root of a number less than 1000 consists of one figure, of any number between 1000 and 1000000, of two places of figures, &c. If then a point be made over every third figure contained in any number, beginning with the units, the number of points will shew the number of places in its cube root.

Ex. 1. Let the cube root of 405224 be required.

$$
\begin{array}{rrl}
 & 40\dot{5}22\dot{4} & (70 + 4 \\
a^3 = & 343000 & \\
\hline
3a^2 = 14700) & 62224 & \text{the first remainder.} \\
\hline
 & 58800 & = 3a^2b \\
 & 3360 & = 3ab^2 \\
 & 64 & = b^3 \\
\hline
 & 62224 & \text{subtrahend.} \\
\hline
 & * \quad * &
\end{array}
$$

By pointing the number according to the direction, it appears that the root consists of two places; let a be the value of the figure in the tens' place, and b of that in the units'. Then a is the greatest number whose cube is contained in 405000, that is, 70; subtract its cube from the whole quantity, and the remainder is 62224; divide this remainder by $3a^2$, or 14700, and the quotient 4, or b, is the second term in the root: then subtract the cube of 74 from the original number, and as the remainder is nothing, 74 is the cube root required. Observe that the cyphers may be omitted in the operation; and that as a^3 was at first subtracted, if from the first remainder $3a^2b + 3ab^2 + b^3$ be taken, the whole cube of $a + b$ will be taken from the original quantity.

159. In extracting the cube root of a *decimal* care must be taken that the decimal places be three, or some multiple of three, before the operation is begun, by annexing cyphers to the right (Art. 41); because there are three times as many decimal places in the cube as there are in the root (Art. 46).

Ex. 2. **Required the cube root of 311897·91.**

$3\dot{1}189\dot{7}\cdot91\dot{0}\,(67\cdot8$
$216\ldots = a^3$
$3a^2 = 108\,..)\ 95897$ first remainder.
$756\,.. = 3a^2b$
$882\,. = 3ab^2$
$343 = b^3$
84763 subtrahend.
$3a^2 = 13467\,..)\ 11134910$ second remainder.

The new value of a is 670, or, omitting the cypher, 67; and $3a$, the new divisor, is 13467 .. hence 8 is the next figure in the root; and

$107736\,.. = 3a^2b$
$12864\,. = 3ab^2$
$512 = b^3$
10902752 subtrahend.
232158 the third remainder.

It appears from the pointing, that there is one decimal place in the root; therefore 67·8 is the root required nearly. If three more cyphers be annexed to the decimal, another decimal place is obtained in the root; and thus approximation may be made to the true root of the proposed number to any required degree of accuracy.

160. Since the first remainder is $3a^2b + 3ab^2 + b^3$, the exact value of b is not obtained by dividing by $3a^2$; and if upon trial the subtrahend be found to be greater than the first remainder, the value assumed for b is too great, and a less number must be tried.

The greater a is with respect to b, the more nearly is the true value obtained by division.

For the first remainder divided by $3a^2$ gives $b + \frac{b^2}{a} + \frac{b^3}{3a^2}$ for the quotient; and if this be adopted for b, the error $= \frac{b^2}{a} + \frac{b^3}{3a^2}$; which for a given value of b is evidently less as a is greater.

161. In extracting the square or cube root of a vulgar fraction the rule stated in Art. 151 may be followed; but it is generally preferable to convert the vulgar fraction into a decimal, and then extract the root.

Thus let the cube root of $5\frac{1}{2}$, or $\frac{11}{2}$ be required.

Now, if the rule of Art. 151 be applied to this case, the cube root of 11, and the cube root of 2, must be found to a certain number of places of decimals, and then the long division of the one root by the other must be effected: whereas, if $5\frac{1}{2}$ be, first of all, converted into a decimal, viz. 5·5, one single extraction of the cube root completes the whole process.

Another method is, to multiply the numerator and denominator by such a quantity as will make the latter a perfect cube, and then apply the rule of Art. 151.

162. In extracting either the square or cube root of any number, when a certain number of figures in the root have been obtained by the common rule, that number may be nearly doubled by division only.

I. The square root of any number may be found by using the common Rule for extracting the square root until *one more than half the number* of digits in the root is obtained; then the rest of the digits in the root may be determined by Division.

For, let N represent the number whose square root, consisting of $2n+1$ digits, is required;

a .... the first $\overline{n+1}$ digits of the root found by the common Rule, with n cyphers annexed;

x .... the remaining part:

so that $\sqrt{N}=a+x$.

Then $N=a^2+2ax+x^2$ (Art. 81);

$$\therefore\ \frac{N-a^2}{2a}=x+\frac{x^2}{2a} \text{ (Arts. 80, 82.)};$$

that is, $N-a^2$, (which is the remainder after $\overline{n+1}$ digits of the root are found) divided by $2a$ will give the rest of the root required, increased by $\frac{x^2}{2a}$. Now, since x contains n digits, x^2 has $2n$ at most (Art. 155). But, by the supposition, $2a$ has $2n+1$ digits; therefore

$$x^2<2a, \text{ or } \frac{x^2}{2a} \text{ is a proper fraction, or} <1;$$

that is, if the quotient of $(N-a^2)\div 2a$ be taken for x, the error is less than 1.

Hence it appears, that if $\overline{n+1}$ digits of a square root are obtained by the common Rule, n digits more may be correctly obtained by Division only.

Ex. Required the square root of 2 to 6 places of decimals.

```
         2·0000...(1·414
         1
        ----
   24) 100
        96
        -----
   281) 400
        281
        ------
  2824) 11900
        11296
       ---------
    2828) 604000(213
           5656
          ------
            3840
            2828
           ------
           10120
            8484
           ------
            1636
```

∴ the root required is 1·414213......

II. When a cube root consists of $\overline{2n+2}$ digits, and $\overline{n+2}$, one more than half the number, have been found by the ordinary rule, the rest can be found by dividing by the trial divisor.

Let $a+b$ be the cube root,

where a consists of $\overline{n+2}$ digits, and n cyphers,

b .... n digits.

$a^3+3a^2b+3ab^2+b^3$ the quantity whose root is required.

Then after a has been found, we have

$$\text{remainder } = 3a^2b+3ab^2+b^3;$$

$$\text{trial divisor } = 3a^2;$$

$$\therefore \text{ quotient } = b+\frac{b^2}{a}+\frac{b^3}{3a^2};$$

If this be adopted as the value of b,

$$\text{the error } = \frac{b^2}{a}+\frac{b^3}{3a^2}.$$

$$\text{Now } b < 10^n, \quad \therefore b^2 < 10^{2n};$$

and a is not less than 10^{2n+1}, ∵ it has $\overline{2n+2}$ digits in all, and 10^{2n+1} is the least number having so many;

$$\therefore \frac{b^2}{a} < \frac{10^{2n}}{10^{2n+1}} < \frac{1}{10}.$$

Also $b^3 < 10^{3n}$,

and a^2 is not less than 10^{4n+2};

$$\therefore \frac{b^3}{3a^2} < \frac{10^{3n}}{3 \times 10^{4n+2}} < \frac{1}{3 \times 10^{n+2}};$$

$$\therefore \text{error} < \frac{1}{10} + \frac{1}{3 \times 10^{n+2}} < 1;$$

that is, the last n figures of the root may be correctly obtained by division only.

REDUCTION OF SURDS.

163. *A rational quantity may be reduced to the form of a given surd by raising it to the power whose root the surd expresses and affixing the radical sign.*

Thus $a = \sqrt{a^2} = \sqrt[3]{a^3}$, &c. and $a + x = (a + x)^{\frac{m}{m}} = \sqrt[m]{(a + x)^m}$.

In the same manner, the form of any surd may be altered; thus $(a + x)^{\frac{1}{2}} = (a + x)^{\frac{2}{4}} = (a + x)^{\frac{3}{6}}$ &c. The quantities are here raised to certain powers, and the roots of those powers are again taken; therefore the values of the quantities are not altered.

164. *The coefficient of a surd may be introduced under the radical sign by first reducing it to the form of the surd, by the last* Art., *and then multiplying according to* Art. 149.

Exs. $a\sqrt{x} = \sqrt{a^2} \times \sqrt{x} = \sqrt{a^2 x}$;

$$ay^{\frac{3}{2}} = (a^2y^3)^{\frac{1}{2}}; \qquad x\sqrt{2a - x} = \sqrt{2ax^2 - x^3};$$

$$a \times (a - x)^{\frac{3}{2}} = \{a^2 \times (a - x)^3\}^{\frac{1}{2}};$$

$$4\sqrt{2} = \sqrt{16 \times 2} = \sqrt{32}.$$

165. *Conversely, any quantity may be made the coefficient of a surd, if every part under the sign be divided by this quantity raised to the power whose root the sign expresses.*

Thus $\sqrt{a^2 - ax} = a^{\frac{1}{2}}\sqrt{a - x}$; $\sqrt{a^3 - a^2x} = a\sqrt{a - x}$;

$$(a^2 - x^2)^{\frac{1}{n}} = a^{\frac{2}{n}}\left(1 - \frac{x^2}{a^2}\right)^{\frac{1}{n}}; \qquad \sqrt{60} = \sqrt{4 \times 15} = 2\sqrt{15};$$

$$\left(\frac{1}{b^2} - \frac{1}{x^2}\right)^{\frac{1}{2}} = \frac{1}{b}\sqrt{1 - \frac{b^2}{x^2}}.$$

ADDITION AND SUBTRACTION OF SURDS.

166. *When surds have the same irrational part, their sum or difference is found by affixing to that irrational part the sum or difference of their coefficients.*

Thus $a\sqrt{x} \pm b\sqrt{x} = (a \pm b)\sqrt{x}$;

$$10\sqrt{3} \pm 5\sqrt{3} = 15\sqrt{3}, \text{ or } 5\sqrt{3}.$$

If the proposed surds have not the same irrational part, they may sometimes be reduced to others which have, by Art. 165. Thus,

Ex. 1. Let the sum of $\sqrt{3a^2b}$ and $\sqrt{3b}$ be required.

Since $\sqrt{3a^2b} = \sqrt{a^2} \times \sqrt{3b} = a\sqrt{3b}$,

$\therefore \sqrt{3a^2b} + \sqrt{3b} = a\sqrt{3b} + \sqrt{3b} = (a+1)\sqrt{3b}$.

Ex. 2. Find the sum of $4a\sqrt[3]{a^3b^4}$, $b\sqrt[3]{8a^6b}$, and $-\sqrt[3]{125a^6b^4}$.

Here $4a\sqrt[3]{a^3b^4} = 4a\sqrt[3]{a^3b^3}.\sqrt[3]{b} = 4a^2b\sqrt[3]{b}$,

$b\sqrt[3]{8a^6b} = b\sqrt[3]{8a^6}.\sqrt[3]{b} = 2a^2b\sqrt[3]{b}$,

$-\sqrt[3]{125a^6b^4} = -\sqrt[3]{125a^6b^3}.\sqrt[3]{b} = -5a^2b\sqrt[3]{b}$;

$\therefore$ the sum required $= a^2b\sqrt[3]{b}$.

If the proposed surds cannot be reduced to others which have the same irrational part, then they must be connected together merely by the signs + and –.

MULTIPLICATION OF SURDS.

167. *If two surds have the same index, their product is found by taking the product of the quantities under the signs and retaining the common index.*

Thus $\sqrt[n]{a} \times \sqrt[n]{b} = a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$ (Art. 149) $= \sqrt[n]{ab}$.

$$\sqrt{2} \times \sqrt{3} = \sqrt{6}; \qquad (a+b)^{\frac{1}{2}} \times (a-b)^{\frac{1}{2}} = (a^2 - b^2)^{\frac{1}{2}};$$

$$\sqrt[3]{a+x} \times \sqrt[3]{a-x} = \sqrt[3]{a^2 - x^2}.$$

If the surds have coefficients, the product of these coefficients must be prefixed.

Thus $a\sqrt{x} \times b\sqrt{y} = ab\sqrt{xy}$.

168. *If the indices of two surds have a common denominator, let the quantities be raised to the powers expressed by their respective numerators, and their product may be found as before.*

Ex. $2^{\frac{3}{2}} \times 3^{\frac{1}{2}} = (2^3)^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 8^{\frac{1}{2}} \times 3^{\frac{1}{2}} = (24)^{\frac{1}{2}}$;

also $(a+x)^{\frac{1}{2}} \times (a-x)^{\frac{3}{2}} = \{(a+x)\,(a-x)^3\}^{\frac{1}{2}}$.

169. *If the indices have not a common denominator, they may be transformed to others of the same value with a common denominator, and their product found as in* Art. 168.

$$\text{Ex.}\quad (a^2-x^2)^{\frac{1}{4}} \times (a-x)^{\frac{1}{2}} = (a^2-x^2)^{\frac{1}{4}} \times (a-x)^{\frac{2}{4}}$$

$$= \{(a^2-x^2) \times (a-x)^2\}^{\frac{1}{4}};$$

again $2^{\frac{1}{2}} \times 3^{\frac{1}{3}} = 2^{\frac{3}{6}} \times 3^{\frac{2}{6}} = (8 \times 9)^{\frac{1}{6}} = (72)^{\frac{1}{6}}$.

170. *If two surds have the same rational quantity under the radical signs, their product is found by making the sum of the indices the index of that quantity.*

Thus $\sqrt[n]{a} \times \sqrt[m]{a} = a^{\frac{1}{n}} \times a^{\frac{1}{m}} = a^{\frac{m}{mn}} \times a^{\frac{n}{mn}} = a^{\frac{m+n}{mn}}$; (see Arts. 149, and 150.)

Ex. $\sqrt{2} \times \sqrt[3]{2} = 2^{\frac{1}{2}} \times 2^{\frac{1}{3}} = 2^{\frac{3}{6}+\frac{2}{6}} = 2^{\frac{5}{6}}$.

DIVISION OF SURDS.

171. *If the indices of two quantities have a common denominator, the quotient of one divided by the other is obtained by raising them respectively to the powers expressed by the numerators of their indices, and extracting that root of the quotient which is expressed by the common denominator.*

For $\dfrac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}} = \left(\dfrac{a}{b}\right)^{\frac{1}{n}}$; and $\dfrac{a^{\frac{m}{n}}}{b^{\frac{p}{n}}} = \dfrac{(a^m)^{\frac{1}{n}}}{(b^p)^{\frac{1}{n}}} = \left(\dfrac{a^m}{b^p}\right)^{\frac{1}{n}}$ (Art. 151.)

Ex. $4^{\frac{1}{2}} \div 2^{\frac{3}{2}} = \left(\dfrac{4}{8}\right)^{\frac{1}{2}} = \dfrac{1}{\sqrt{2}}$; $\left(\dfrac{p}{q}\right)^{\frac{1}{m}} \div \left(\dfrac{r}{s}\right)^{\frac{2}{m}} = \left(\dfrac{ps^2}{qr^2}\right)^{\frac{1}{m}}$.

172. *If the indices have not a common denominator, reduce them to others of the same value with a common denominator, and proceed as before.*

$$\text{Ex.}\quad (a^2-x^2)^{\frac{1}{2}} \div (a^3-x^3)^{\frac{1}{3}} = (a-x^2)^{\frac{3}{6}} \div (a^3-x^3)^{\frac{2}{6}}$$

$$= \left\{\frac{(a^2-x^2)^3}{(a^3-x^3)^2}\right\}^{\frac{1}{6}}.$$

173. *If two surds have the same rational quantity under the radical signs, their quotient is obtained by making the difference of the indices the index of that quantity.*

Thus $\sqrt[n]{a} \div \sqrt[m]{a}$, or $a^{\frac{1}{n}}$ divided by $a^{\frac{1}{m}}$, or $a^{\frac{m}{mn}}$ divided by $a^{\frac{n}{mn}}$, that is $\dfrac{a^{\frac{m}{mn}}}{a^{\frac{n}{mn}}}$, is equal to $a^{\frac{m-n}{mn}}$; because these quantities, raised to the power mn, produce equal results $\dfrac{a^m}{a^n}$ and a^{m-n}.

Ex. $2^{\frac{1}{2}} \div 2^{\frac{1}{3}} = 2^{\frac{3}{6}} \div 2^{\frac{2}{6}} = 2^{\frac{1}{6}}$.

INVOLUTION AND EVOLUTION OF SURDS.

174. *Any power of a surd is found by multiplying the fractional index of the surd by the number which expresses the power.*

For $(\sqrt[n]{a})^m = (a^{\frac{1}{n}})^m = a^{\frac{1}{n}+\frac{1}{n}+\ldots\text{to } m \text{ terms}} = a^{\frac{m}{n}}$.

175. *Any root of a surd is found by dividing the fractional index of the surd by the number which expresses the root.*

Thus $\sqrt[m]{(\sqrt[n]{a})} = \sqrt[m]{a^{\frac{1}{n}}} = a^{\frac{1}{mn}}$; because each of these quantities raised to the m^{th} power will produce $a^{\frac{1}{n}}$.

It will be seen that the rules hitherto required for the management of surds are simply those which apply to quantities raised to powers expressed by *Fractional Indices*.

TRANSFORMATION OF SURDS.

176. *Having given a quantity containing quadratic surds, to find another quantity which, multiplied into the former, shall produce a rational result.*

1. If the given quantity be a simple surd, as $3\sqrt{a}$, the multiplier required is $\sqrt{a}$, which gives the product $3a$, a rational quantity.

2. If the given quantity be a binomial surd, as $\sqrt{a}+\sqrt{b}$, then the multiplier required is $\sqrt{a}-\sqrt{b}$, and the product is $a-b$.

3. If the quantity be a trinomial, as $\sqrt{a}+\sqrt{b}+\sqrt{c}$, first multiply by $\sqrt{a}+\sqrt{b}-\sqrt{c}$, which gives $(\sqrt{a}+\sqrt{b})^2-(\sqrt{c})^2$, or $a+b-c+2\sqrt{ab}$. Next multiply by $a+b-c-2\sqrt{ab}$, and the product is $(a+b-c)^2-4ab$. Therefore the multiplier required is $(\sqrt{a}+\sqrt{b}-\sqrt{c}) \times (a+b-c-2\sqrt{ab})$.

The use of this proposition is to enable us without much labour to find the values of fractions which have irrational denominators. Thus, suppose the actual value of $\frac{1}{\sqrt{3}+\sqrt{2}}$ were required to 7 places of decimals; if we were to proceed to extract the square roots of 2 and 3, and divide 1 by the sum of those roots, the operation would be long and troublesome. But if we first multiply the numerator and denominator by $\sqrt{3}-\sqrt{2}$, the fraction becomes $\frac{\sqrt{3}-\sqrt{2}}{3-2}$, or $\sqrt{3}-\sqrt{2}$, and its value is not altered; we have simply then to extract the square roots of 3 and 2 and subtract the one from the other, by which the long division is entirely avoided.

So also $$\frac{\sqrt[3]{2}}{\sqrt[3]{3}}=\frac{\sqrt[3]{2}\times\sqrt[3]{3}\times\sqrt[3]{3}}{(\sqrt{3})^3}=\frac{\sqrt[3]{18}}{3},$$

and $$\frac{\sqrt[r]{a}}{\sqrt[r]{b}}=\frac{a^{\frac{1}{r}}\times b^{\frac{r-1}{r}}}{b^{\frac{1}{r}}\times b^{\frac{r-1}{r}}}=\frac{\sqrt[r]{ab^{r-1}}}{b};$$

each of which fractions is thus much simplified for purposes of calculation.

Again, $\frac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}=\frac{1}{\sqrt[3]{x^2}+\sqrt[3]{ax}+\sqrt[3]{a^2}}$; but the former quantity is in a simpler form than the latter.

More generally,

177. *To find the multiplier which will rationalize any binomial, having one or both of its terms irrational.*

1st. Let $x+y$ represent the binomial, x and y being, one or both, irrational, and let m be such a number that x^m and y^m are both rational, that is, let m be the Least Com. Mult. of the denominators of the fractional indices of the binomial; then since

$$x^m \pm y^m = (x+y)\,.\,(x^{m-1}-x^{m-2}y+\ldots\mp xy^{m-2}\pm y^{m-1}),$$

where the upper or lower sign is to be taken, according as m is odd or even, the rationalizing multiplier required is

$$x^{m-1}-x^{m-2}y+\ldots\mp xy^{m-2}\pm y^{m-1}.$$

2d. Let $x-y$ be the binomial, and m as before, then since $x^m-y^m=(x-y)\,.\,(x^{m-1}+x^{m-2}y+\ldots+xy^{m-2}+y^{m-1})$, (Art. 99. Ex. 6.) the rationalizing multiplier is

$$x^{m-1}+x^{m-2}y+\ldots+xy^{m-2}+y^{m-1}.$$

Ex. Find the multiplier which will rationalize $\sqrt{5}-\sqrt{6}$, or $5^{\frac{1}{2}}-6^{\frac{1}{3}}$.

Here $m=6$, the Least Com. Mult. of 2 and 3,

$$\therefore \text{mult}^{\text{r}}.\ \text{req}^{\text{d}}. = (\sqrt{5})^5 + (\sqrt{5})^4 \times \sqrt[3]{6} + (\sqrt{5})^3 \times (\sqrt[3]{6})^2 + (\sqrt{5})^2 \times (\sqrt[3]{6})^3 + \sqrt{5} \times (\sqrt[3]{6})^4 + (\sqrt[3]{6})^5$$

$$= 25\sqrt{5} + 25\sqrt[3]{6} + 5\sqrt{5} \times (\sqrt[3]{6})^2 + 30 + 6\sqrt{5} \times \sqrt[3]{6} + 6(\sqrt[3]{6})^2.$$

178. *The square root of a quantity cannot be partly rational and partly a quadratic surd.*

If possible, let $\sqrt{n} = a + \sqrt{m}$; then, by squaring these equal quantities, $n = a^2 + 2a\sqrt{m} + m$, (Art. 81); and $2a\sqrt{m} = n - a^2 - m$, (Art. 80); therefore $\sqrt{m} = \dfrac{n - a^2 - m}{2a}$, (Art. 82), a rational quantity, which is contrary to the supposition.

179. *If any two quantities, partly rational and partly quadratic surds, be equal to one another, the rational parts of the two are equal, and also the irrational parts.*

Let $x + \sqrt{y} = a + \sqrt{b}$, then $x = a$, and $\sqrt{y} = \sqrt{b}$; for if x be not equal to a, let $x = a + m$; then $a + m + \sqrt{y} = a + \sqrt{b}$, or $m + \sqrt{y} = \sqrt{b}$; that is, $\sqrt{b}$ is partly rational and partly a quadratic surd, which is impossible (Art. 178); therefore $x = a$, and consequently also $\sqrt{y} = \sqrt{b}$.

180. *If two quadratic surds $\sqrt{\mathrm{x}}$ and $\sqrt{\mathrm{y}}$ cannot be reduced to others which have the same irrational part, their product is irrational.*

If possible, let $\sqrt{xy} = rx$, where r is a whole number or a fraction. Then $xy = r^2x^2$ (Art. 81), and $y = r^2x$ (Art. 82); therefore $\sqrt{y} = r\sqrt{x}$, that is, $\sqrt{y}$ and $\sqrt{x}$ may be so reduced as to have the same irrational part, which is contrary to the supposition.

181. *One quadratic surd, $\sqrt{\mathrm{x}}$, cannot be made up of two others, $\sqrt{\mathrm{m}}$ and $\sqrt{\mathrm{n}}$, which have not the same irrational part.*

If possible, let $\sqrt{x} = \sqrt{m} + \sqrt{n}$; then by squaring these equal quantities, $x = m + n + 2\sqrt{mn}$, and $x - m - n = 2\sqrt{mn}$, a rational quantity equal to an irrational one; which is absurd.

182. *The square root of a binomial, one of whose terms s a quadratic surd, and the other rational, may sometimes be expressed by a binomial, one or both of whose terms are quadratic surds.*

Since $(\sqrt{x} \pm \sqrt{y})^2 = x + y \pm 2\sqrt{xy}$, $\sqrt{x + y \pm 2\sqrt{xy}} = \sqrt{x} \pm \sqrt{y}$; hence f any proposed binomial surd can be put under the form $(x + y) \pm 2\sqrt{xy}$ ts square root is at once found by inspection to be $\sqrt{x} \pm \sqrt{y}$. Now, to proceed with any proposed case, take the term which contains the surd, and if it can be put into factors of the form $2\sqrt{x} \times \sqrt{y}$ in more ways than one, take that pair of factors for which the sum of x and y is equal to the whole of the term in the proposed binomial which is rational. Having thus found x and y, the square root required is $\sqrt{x} \pm \sqrt{y}$, + or – according as the ign of the *surd* in the proposed binomial is + or –.

Ex. 1. Required the square root of $3 + 2\sqrt{2}$.

$$\text{Here } 2\sqrt{2} = 2\sqrt{2} \times \sqrt{1};$$

$$\text{also } 2 + 1 = 3, \text{ the rational term};$$

$$\therefore \text{ the root required is } 1 + \sqrt{2}.$$

Ex. 2. Required the square root of $7 - 2\sqrt{10}$.

$$\text{Here } 2\sqrt{10} = 2\sqrt{5} \times \sqrt{2},$$

$$\text{also } 5 + 2 = 7, \text{ the rational term};$$

$$\therefore \text{ root required is } \sqrt{5} - \sqrt{2}.$$

Ex. 3. Required the square root of $11 - 6\sqrt{2}$.

Here $6\sqrt{2} = 2\sqrt{18} = 2\sqrt{9} \,.\, \sqrt{2}$, or $2\sqrt{6} \,.\, \sqrt{3}$; of which the former nswers the condition required, viz. $9 + 2 = 11$, the rational term;

$$\therefore \text{ the root required is } \sqrt{9} - \sqrt{2}, \text{ that is, } 3 - \sqrt{2}.$$

Ex. 4. Required the square root of $2x + 2\sqrt{x^2 - 1}$.

$$\text{Here } 2\sqrt{x^2 - 1} = 2\sqrt{x + 1} \times \sqrt{x - 1},$$

$$\text{also } x + 1 + x - 1 = 2x, \text{ the rational term};$$

$$\therefore \text{ root required is } \sqrt{x + 1} + \sqrt{x - 1}.$$

Ex. 5. Required the square root of $7 + \sqrt{13}$.

$$\text{Here } \sqrt{13} = 2\sqrt{\frac{13}{4}} = 2\sqrt{\frac{13}{2}} \times \sqrt{\frac{1}{2}}, \text{ also } \frac{13}{2} + \frac{1}{2} = 7;$$

$$\therefore \text{ root required is } \frac{\sqrt{13} + 1}{\sqrt{2}}.$$

A more general method of extracting the roots of irrational binomials will be given hereafter.

IMAGINARY OR IMPOSSIBLE QUANTITIES.

183. If an expression appear under the form $\sqrt{-a}$, this indicates an impossibility; for it signifies the square root of a negative quantity, which has no existence, since there is no quantity, positive or negative, which, being multiplied by itself, gives a negative product. It is evident, therefore, that if such symbols are admitted into calculations, they may require special Rules for themselves, and some care in the application of those Rules. And it must be borne in mind, that they are mere symbols, and not expressions of quantity.

We must observe especially, if we do meet with such *imaginary* quantities, that the product of $\sqrt{-a}\times\sqrt{-a}$ does not follow the rule given in Art. 167, but that it is $-a$, because it is that quantity whose square root is $\sqrt{-a}$. Also similarly $\sqrt{-a}\times\sqrt{-b}$ is not $\sqrt{-a\times-b}$, or $\sqrt{ab}$, but it is $\sqrt{a}\times\sqrt{-1}\times\sqrt{b}\times\sqrt{-1}$, or $\sqrt{ab}\times(\sqrt{-1})^2$, that is, $-\sqrt{ab}$[*].

To avoid mistakes in operating upon *imaginary* quantities, as $\sqrt{-a}$, $\sqrt{-b}$, &c., it will be best in all cases to substitute for them their equivalents $\sqrt{a}\cdot\sqrt{-1}$, $\sqrt{b}\cdot\sqrt{-1}$, &c., and to bear in mind that $(\sqrt{-1})^2=-1$, $(\sqrt{-1})^3=-\sqrt{-1}$, $(\sqrt{-1})^4=+1$, &c.

Ex. $(x-a+\sqrt{-b^2})\,(x-a-\sqrt{-b^2})=(x-a+b\sqrt{-1})\,(x-a-b\sqrt{-1})$

$$=x^2-2ax+a^2+b^2.$$

EQUATIONS.

184. If one quantity be equal to another, or to nothing, and this equality be expressed algebraically, it constitutes an *Equation*. Thus $x-a=b-x$ is an *Equation*, of which $x-a$ forms one *side*, and $b-x$ the other.

In this equation it is asserted, that a certain *unknown quantity* (x) is so connected with two known quantities (a and b), that it exceeds the one (a) by as much as it falls short of the other (b).

Again, $x^2+x-20=0$ is an *Equation*, which asserts that a certain unknown number added to its square is equal to 20.

An equality which admits of no question, as $x+a=x+a$, or $2x+3x=5x$, is not an "Equation" strictly speaking, but is called an "Identity." An Identity is therefore satisfied by *any value whatever* of the unknown quantity; whereas in "*Equations*" the unknown quantities ve *particular values,* which alone, and none other, will permit the pressed equality to subsist.—To find these values is to "*solve*" the quations, and forms an important part of the business of Algebra. hese values are sometimes called the "*Roots*" of the Equations, and are id to *satisfy* them. Thus, if $2x=6$ be the Equation, $x=3$, and can nothing else; and $x=3$ is called its *solution.* Again, if $x^2=4$, we ow that $x=2$ or -2; and $2, -2$, are called the *Roots* of the equation $= 4$.

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[*] The student is recommended to have as little as possible to do with *imaginary* quantities, that is, with quantities which have no meaning either as to *number* or *magnitude*. He need not wonder that difficulties are likely to be introduced by the use of them, when he considers that $\sqrt{-a}$ signifies an operation to be performed which is absolutely impossible. Any discussion upon the interpretation which *may* be given to such symbols, and the uses to which they may be applied, would be quite out of place in an Elementary Treatise like the present.

185. *When an equation is cleared of fractions and surds,* if contain the first power only of an unknown quantity, it is called *Simple Equation*, or an equation of one dimension; if the *uare* of the unknown quantity be in any term, it is called a *uadratic*, or an equation of two dimensions; if the *Cube* of e unknown quantity appear, it is called a *Cubic Equation*; if e fourth power, a *Biquadratic*; and in general, if the index the highest power of the unknown quantity be n, it is called *equation of* n *dimensions.*

SIMPLE EQUATIONS.

186. RULE I. *In any equation quantities may be transsed from one side to the other, if their signs be changed, and e two sides will still be equal.*

For let $x + 10 = 15$; then by subtracting 10 from each side, rt. 80), $x + 10 - 10 = 15 - 10$, or $x = 15 - 10$.

Let $x - 4 = 6$; by adding 4 to each side, (Art. 79),

$$x - 4 + 4 = 6 + 4, \text{ or } x = 6 + 4.$$

If $x - a + b = y$; adding $a - b$ to each side,

$$x - a + b + a - b = y + a - b; \text{ or } x = y + a - b.$$

187. COR. Hence, if the signs of *all* the terms on each side changed, the two sides will still be equal.

$$\text{Let } x - a = b - 2x;$$

$$\text{by transposition, } -b + 2x = -x + a;$$

$$\text{or } a - x = 2x - b.$$

188. RULE II. *If every term on each side be multiplied by e same quantity, the results will be equal* (Art. 81).

189. **Cor.** An equation may be cleared of fractions, by multiplying every term successively by the denominators of those fractions.

$$\text{Let } 3x + \frac{5x}{4} = 34;$$

multiplying by 4, $12x + 5x = 136$.

An equation may be cleared of fractions at once, by multiplying both sides by the product of all the denominators, or by any quantity which is a multiple of them all.

$$\text{Let } \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = 13;$$

multiplying by $2 \times 3 \times 4$, $3 \times 4 \times x + 2 \times 4 \times x + 2 \times 3 \times x = 2 \times 3 \times 4 \times 13$,

or $12x + 8x + 6x = 312$; that is, $26x = 312$.

If the Least Common Multiple of the denominators be made use of, the equation will be *in the lowest terms.*

Thus, if each side of the last equation be multiplied by 12, which is the Least Com. Mult. of 2, 3, and 4, the equation will become

$$\frac{12x}{2} + \frac{12x}{3} + \frac{12x}{4} = 156;$$

or $6x + 4x + 3x = 156$; that is, $13x = 156$.

190. **Rule III.** *If each side of an equation be divided by the same quantity, the results will be equal.* (Art. 82.)

Let $17x = 136$; then $x = \frac{136}{17} = 8$.

191. **Rule IV.** *If each side of an equation be raised to the same power, the results will be equal.* (Art. 81.)

Let $x^{\frac{1}{2}} = 9$; then $x = 9 \times 9 = 81$.

Also, if the same root be extracted on both sides, the results will be equal.

Let $x = 81$; then $x^{\frac{1}{2}} = \pm 9$ (Art. 143).

192. **Rule** V. *To clear an equation of surds.*

An equation may be cleared of a surd by transposing the terms so that the surd shall form one side, and the rational quantities the other, and then raising both sides to that power whch will rationalize the surd.

Thus, if $\sqrt{a+x}-b=c$, by transposition $\sqrt{a+x}=b+c$, and $a+x=(b+c)^2$. (Art. 81).

If the equation contain *two* surds, connected by + or −, then the same operation must be repeated for the second surd.

$$\begin{aligned} &\text{Thus, if } \sqrt{a+x}+\sqrt{x}=b,\\ &\text{by transp.}\quad \sqrt{a+x}=b-\sqrt{x},\\ &\text{squaring,}\quad a+x=b^2-2b\sqrt{x}+x,\\ &\text{by transp.}\quad 2b\sqrt{x}=b^2-a,\\ &\text{squaring,}\quad 4b^2x=(b^2-a)^2, \end{aligned}$$

an equation in which the surds do not appear.

193. *A "simple equation" can have only one solution; that is, there can be but one value of the unknown quantity which satisfies it.*

For every "simple equation" with respect to the unknown quantity x can be reduced to the form $ax+b=0$. Now, if possible, let there be two values of x which satisfy this equation, viz. α and β;

$$\begin{aligned} \text{then}\quad & a\alpha+b=0,\\ \text{and}\quad & a\beta+b=0;\\ \therefore \text{subtracting,}\quad & a\alpha-a\beta=0,\\ \text{or}\quad & a(\alpha-\beta)=0. \end{aligned}$$

But a cannot be equal to 0, for then the proposed equation would be no equation at all with respect to x, therefore $\alpha-\beta=0$, or $\alpha=\beta$; that is, α and β cannot be different values; or there is only one value of x which satisfies the equation.

194. *To find the value of an unknown quantity in a simple equation.*

Let the equation first be cleared of fractions and surds*, (Arts. 189, 192.) then transpose all the terms which involve the unknown quantity to one side of the equation, and the known quantities to the other (Art 186); divide both sides by the coefficient, or sum

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* It should be borne in mind that this is required to be done only when *the unknown quantity* is found in a fraction or surd. Thus it will not be necessary in such equations as the following:—

$$2x+\frac{4}{5}=x+\frac{6}{7}.$$

$$nx+\sqrt{a}=mx+\sqrt{b}.$$

of the coefficients, of the unknown quantity, (Art. 190) and the val
required is obtained.

Ex. 1. To find the value of x in the equation $3x - 5 = 23 - x$

$$\text{By transp.}\quad 3x + x = 23 + 5, \quad \text{(Art. 186.)}$$

$$\text{or}\quad 4x = 28;$$

$$\text{by division}\quad x = \frac{28}{4} = 7. \quad \text{(Art 190.)}$$

Ex. 2. Let $x + \frac{x}{2} - \frac{x}{3} = 4x - 17$; required x.

$$\text{Mult. by 2,}\quad 2x + x - \frac{2x}{3} = 8x - 34,$$

$$\text{Mult. by 3,}\quad 6x + 3x - 2x = 24x - 102 \quad \text{(Art. 189.),}$$

$$\text{by transp.}\quad 6x + 3x - 2x - 24x = -102,$$

$$\text{or}\quad -17x = -102,$$

$$17x = 102 \quad \text{(Art. 187.)};$$

$$\therefore x = \frac{102}{17} = 6.$$

Ex. 3. $\frac{1}{a} + \frac{b}{x} = c$; required x.

$$\text{Mult. by } a,\quad 1 + \frac{ba}{x} = ca,$$

$$\text{Mult. by } x,\quad x + ba = cax,$$

$$\text{By transp.}\quad x - cax = -ba,$$

$$cax - x = ba \quad \text{(Art. 187.)};$$

$$(ca - 1)\,x = ba;$$

$$\therefore x = \frac{ba}{ca - 1}.$$

Ex. 4. $5 - \frac{x + 4}{11} = x - 3$; required x.

$$55 - x - 4^* = 11x - 33,$$

$$55 - 4 + 33 = 11x + x,$$

$$84 = 12x;$$

$$\therefore x = \frac{84}{12} = 7.$$

Ex. 5. $x + \frac{3x - 5}{2} = 12 - \frac{2x - 4}{3}$; required x.

$$2x + 3x - 5 = 24 - \frac{4x - 8}{3},$$

$$6x + 9x - 15 = 72 - 4x + 8^*,$$

$$6x + 9x + 4x = 72 + 8 + 15,$$

$$19x = 95;$$

$$\therefore\ x = \frac{95}{19} = 5.$$

Ex. 6. $\frac{7x+8}{8} - \frac{9x-12}{16} = \frac{3x+1}{10} - \frac{29-8x}{20}$; required x.

Here the Least Com. Mult. of the denominators is 80, (Art. 23); herefore, multiplying both sides by this number, (Art. 189),

$$70x + 80 - 45x + 60 = 24x + 8 - 116 + 32x^*,$$

$$70x - 45x - 24x - 32x = 8 - 116 - 80 - 60,$$

$$-31x = -248,$$

$$\text{or } 31x = 248;$$

$$\therefore\ x = \frac{248}{31} = 8.$$

Ex. 7. $\frac{1}{14}\left(3x + \frac{2}{3}\right) - \frac{1}{7}\left(4x - 6\tfrac{2}{3}\right) = \frac{1}{2}(5x - 6)$; required x.

Mult. by 14, $3x + \frac{2}{3} - 8x + 12\tfrac{4}{3} = 35x - 42,$

$$42 + 12\tfrac{4}{3} + \frac{2}{3} = 35x + 8x - 3x,$$

$$42 + 12 + 2 = 40x,\ \because \frac{4}{3} + \frac{2}{3} = 2,$$

$$40x = 56;$$

$$\therefore\ x = \frac{56}{40} = \frac{7}{5} = 1\tfrac{2}{5}.$$

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* See Art. 87, bearing in mind that the line which separates the numerator and denomitor of a fraction serves as a *vinculum* for both.

Ex. 8. $\frac{x-4\frac{2}{3}}{3}-\frac{2x-3\frac{2}{3}}{4}=\frac{3}{2}\left\{x-\frac{x-1\frac{1}{3}}{2}\right\}+\frac{4x}{3}\left\{x-3-\frac{\overline{x-1}\,.\,\overline{x-2}}{x}\right\}$ required x.

Mult. by 12, the L. C. M. of 3, 4, and 2,

$$4x-16\tfrac{2}{3}-6x+11=18\,.\,\frac{x+1\frac{1}{2}}{2}+16x\,.\,\frac{x^2-3x-(x^2-3x+2)}{x}$$

$$=9x+13\tfrac{1}{2}-32,$$

$$32+11-16\tfrac{2}{3}-13\tfrac{1}{2}=9x+6x-4x,$$

$$11x=14-\frac{8}{3}-\frac{1}{2}=12-\frac{2}{3}-\frac{1}{2}=\frac{65}{6};$$

$$\therefore\ x=\frac{65}{66}.$$

Ex. 9. $\frac{8x+5}{14}+\frac{7x-3}{6x+2}=\frac{16x+15}{28}+\frac{2\frac{1}{4}}{7}$; required x*.

Mult. by 28, $16x+10+\frac{196x-84}{6x+2}=16x+15+9,$

$$\frac{196x-84}{6x+2}=24-10=14,$$

$$196x-84=84x+28,$$

$$112x=112;$$

$$\therefore\ x=1.$$

Ex. 10. $\frac{ad-bc}{d(c+dx)}+\frac{b}{d}=\frac{2a-bx}{c+dx}$; required x.

Mult. by d, $\frac{ad-bc}{c+dx}+b=\frac{2ad-bdx}{c+dx},$

$$b=\frac{2ad-bdx-(ad-bc)}{c+dx}$$

$$=\frac{ad-bdx+bc}{c+dx},$$

$$bc+bdx=ad-bdx+bc,$$

$$2bdx=ad;$$

$$\therefore\ x=\frac{ad}{2bd}=\frac{a}{2b}.$$

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* In cases like this, which have one or more *compound* denominators involving t unknown quantity, it will usually be found convenient to clear the equation of the *simp* denominators first, leaving the fractions with compound denominators to be dealt with afte wards, when the equation has been reduced to fewer terms.

Ex. 11. $\sqrt{a+x} + \sqrt{a-x} = 2\sqrt{x}$; required x.

$$\sqrt{a+x} = 2\sqrt{x} - \sqrt{a-x},$$

$$a + x = 4x - 4\sqrt{ax - x^2} + a - x,$$

$$4\sqrt{ax - x^2} = 2x,$$

$$2\sqrt{ax - x^2} = x,$$

$$4ax - 4x^2 = x^2,$$

$$4ax = 5x^2,$$

$$4a = 5x;$$

$$\therefore\ x = \frac{4a}{5}.$$

Ex. 12. $\sqrt[m]{a+x} = \sqrt[2m]{x^2 + 5ax + b^2}$; required x.

Raising both sides to the $2m^{\text{th}}$ power, we have

$$(a+x)^2 = x^2 + 5ax + b^2,$$

$$\text{or } a^2 + 2ax + x^2 = x^2 + 5ax + b^2,$$

$$3ax = a^2 - b^2;$$

$$\therefore\ x = \frac{a^2 - b^2}{3a}.$$

195. A very useful formula in solving equations is the following:— a, b, c, d be any quantities whatever, and

$$\text{if } \frac{a}{b} = \frac{c}{d}, \text{ then } \frac{a+b}{a-b} = \frac{c+d}{c-d}; \text{ and } \frac{a-b}{a+b} = \frac{c-d}{c+d}.$$

To prove this,

$$\frac{a}{b} = \frac{c}{d};$$

$$\therefore\ \frac{a}{b} + 1 = \frac{c}{d} + 1,$$

$$\text{or } \frac{a+b}{b} = \frac{c+d}{d}.$$

$$\text{Again } \frac{a}{b} - 1 = \frac{c}{d} - 1,$$

$$\text{or } \frac{a-b}{b} = \frac{c-d}{d};$$

$$\therefore\ \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d},$$

$$\text{or } \frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

Cor. Hence also $\dfrac{a-b}{a+b}=\dfrac{c-d}{c+d}$.

Obs. In the application of this formula to cases in which one side of the equation is *a whole number*, the whole number must be considered as a fraction with 1 for its denominator.

It is also useful to express the formula in language, viz. that, "if one fraction be equal to another, the sum of the numerator and denominator divided by their difference, or the difference divided by their sum, for one fraction, is equal to the same expression for the other fraction.

Ex. 1. $\dfrac{x+2}{x-2}=\dfrac{7}{5}$; required the value of x.

By the formula, $$\frac{2x}{4}=\frac{12}{2},$$

$$\text{or } \frac{x}{2}=6;$$

$$\therefore x=12.$$

Ex. 2. Given $\dfrac{\sqrt{a}+\sqrt{a-x}}{\sqrt{a}-\sqrt{a-x}}=\dfrac{1}{a}$; required x.

By formula, $$\frac{2\sqrt{a}}{2\sqrt{a-x}}=\frac{1+a}{1-a},$$

$$\text{or } \sqrt{\frac{a}{a-x}}=\frac{1+a}{1-a};$$

$$\text{squaring, } \frac{a}{a-x}=\left(\frac{1+a}{1-a}\right)^2,$$

$$\text{or } \frac{a-x}{a}=\left(\frac{1-a}{1+a}\right)^2,$$

$$\text{or } 1-\frac{x}{a}=\left(\frac{1-a}{1+a}\right)^2;$$

$$\frac{x}{a}=1-\frac{1-2a+a^2}{1+2a+a^2}=\frac{4a}{(1+a)^2};$$

$$\therefore x=\frac{4a^2}{(1+a)^2}=\left(\frac{2a}{1+a}\right)^2.$$

Ex. 3. Given $\dfrac{\sqrt[3]{x+1}-\sqrt[3]{x-1}}{\sqrt[3]{x+1}+\sqrt[3]{x-1}}=\dfrac{1}{2}$; required x.

By formula, $$\frac{\sqrt[3]{x+1}}{\sqrt[3]{x-1}}=\frac{3}{1};$$

$$\text{cubing,}\quad \frac{x+1}{x-1}=27,$$

$$\text{by formula again,}\quad x=\frac{28}{26}=1\tfrac{1}{13}.$$

N.B. As a general Rule it is not advisable to apply the formula when the unknown quantity occurs on *both* sides of the equation; for if the one side be simplified by it, the other side will often be rendered more complex, and so nothing be gained. Thus no good purpose will be served by applying the formula to the equation

$$\frac{a+x}{a-x}=\frac{x}{a}.$$

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196. If there be two *independent* simple equations involving two unknown quantities, they may be reduced to one which involves only one of the unknown quantities, by any of the following methods:—

First method. In either equation, find the value of one of the unknown quantities in terms of the other and known quantities, and for it substitute this value in the other equation, which will then only contain one unknown quantity, whose value may be found by the rules before laid down.

Ex. $\left.\begin{aligned} x+y&=10\\ 2x-3y&=5\end{aligned}\right\}$; to find x and y.

From the first equation $x=10-y$; hence $2x=20-2y$, by substitution in the second $20-2y-3y=5$,

$$20-5=2y+3y,$$

$$15=5y;$$

$$\therefore y=\frac{15}{5}=3.$$

Hence also $x=10-y=10-3=7$.

Second method. Find an expression for one of the unknown quantities in each equation; put these expressions equal to each other, and from the resulting equation the other unknown quantity may be found.

Ex. $\left.\begin{aligned} x+y&=a\\ bx+cy&=de\end{aligned}\right\}$; to find x and y.

From the first equation $x = a - y$,

from the second $bx = de - cy$, and $x = \dfrac{de - cy}{b}$;

$$\therefore\ a - y = \frac{de - cy}{b},$$

$$ba - by = de - cy,$$

$$cy - by = de - ba,$$

$$(c - b)\,y = de - ba;$$

$$\therefore\ y = \frac{de - ba}{c - b}.$$

Also $x = a - y$;

$$= a - \frac{de - ba}{c - b}$$

$$= \frac{ca - ba - de + ba}{c - b}$$

$$= \frac{ca - de}{c - b}.$$

Third method. If either of the unknown quantities have the same coefficient in both equations, it may be exterminated by subtracting or adding the equations, according as the sign of the unknown quantity, in the two cases, is the same or different.

Ex. $\left.\begin{aligned} x + y &= 15 \\ x - y &= 7 \end{aligned}\right\}$; to find x and y.

By addition $2x = 22$, $\therefore\ x = 11$.

By subtraction $2y = 8$, $\therefore\ y = 4$.

If the coefficients of the unknown quantity to be exterminated be *different*, multiply the terms of the first equation by the coefficient of the unknown quantity in the second, and the terms of the second equation by the coefficient of the same unknown quantity in the first; then add, or subtract, the resulting equations, as in the former case.

This is the method in most general use.

Ex. 1. $$\left.\begin{aligned}3x - 5y &= 13\\ 2x + 7y &= 81\end{aligned}\right\}; \text{ to find } x \text{ and } y.$$

Multiply the terms of the first equation by 2, and the terms of the second by 3,

$$\left.\begin{aligned}\text{then}\quad 6x - 10y &= 26\\ \text{and}\quad 6x + 21y &= 243\end{aligned}\right\}$$

By subtraction $31y = 217$;

$$\therefore\ y = \frac{217}{31} = 7.$$

Also $3x - 5y = 13$,

or $3x - 35 = 13$,

$$3x = 13 + 35 = 48;$$

$$\therefore\ x = \frac{48}{3} = 16.$$

Ex. 2. $$\left.\begin{aligned}ax + by &= c\\ mx - ny &= d\end{aligned}\right\}; \text{ to find } x \text{ and } y.$$

From the first $max + mby = mc$,

from the second $max - nay = ad$,

by subtraction $(mb + na)y = mc - ad$;

$$\therefore\ y = \frac{mc - ad}{mb + na}.$$

Again $nax + nby = nc$,

$mbx - nby = bd$;

by addition $(na + mb)x = nc + bd$,

$$\therefore\ x = \frac{nc + bd}{na + mb}.$$

Ex. 3. $$\left.\begin{aligned}\frac{3x - 5y}{2} + 3 &= \frac{2x + y}{5}\\ 8 - \frac{x - 2y}{4} &= \frac{x}{2} + \frac{y}{3}\end{aligned}\right\}; \text{ to find } x \text{ and } y.$$

From the first $15x - 25y + 30 = 4x + 2y,$

$$15x - 4x - 25y - 2y = -30,$$

$$11x - 27y = -30.$$

From the second $96 - 3x + 6y = 6x + 4y,$

$$96 = 6x + 3x + 4y - 6y;$$

$$\left.\begin{aligned} &9x - 2y = 96 \\ \text{and}\quad &11x - 27y = -30 \end{aligned}\right\}$$

$$\left.\begin{aligned} \text{hence}\quad &99x - 22y = 1056 \\ \text{and}\quad &99x - 243y = -270 \end{aligned}\right\}$$

$$221y = 1326;$$

$$\therefore\ y = \frac{1326}{221} = 6.$$

Also $9x - 2y = 96,$

$$9x - 12 = 96,$$

$$9x = 96 + 12 = 108;$$

$$\therefore x = \frac{108}{9} = 12.$$

197. If there be three *independent* simple equations, and three unknown quantities, reduce two of the equations to one, containing only two of the unknown quantities, by the preceding rules; then reduce the third equation and either of the former to one, containing the same two unknown quantities; and from the two equations thus obtained the unknown quantities which they involve may be found. The third quantity may be found by substituting their values in any of the proposed equations.

Ex. $$\left.\begin{aligned} 2x + 3y + 4z &= 16 \\ 3x + 2y - 5z &= 8 \\ 5x - 6y + 3z &= 6 \end{aligned}\right\};\ \text{to find } x,\ y \text{ and } z.$$

From the 1^{st} two equ$^{\text{ns}}$. $$\left.\begin{aligned} 6x + 9y + 12z &= 48, \\ 6x + 4y - 10z &= 16, \end{aligned}\right\}$$

by subtr. $5y + 22z = 32.$

From the 1st and 3rd $$\left.\begin{aligned}10x + 15y + 20z &= 80\\ 10x - 12y + 6z &= 12\end{aligned}\right\}$$

$$\begin{aligned}\text{by subtr.}\quad &\left.\begin{aligned}27y + 14z &= 68\\ \text{and}\quad 5y + 22z &= 32\end{aligned}\right\}\\ \text{hence}\quad &\left.\begin{aligned}135y + 70z &= 340\\ \text{and}\quad 135y + 594z &= 864\end{aligned}\right\}\\ \text{by subtr.}\quad & 524z = 524;\\ & \therefore\ z = 1.\end{aligned}$$

$$\begin{aligned}\text{Also}\quad 5y + 22z &= 32,\\ \text{that is,}\quad 5y + 22 &= 32,\\ 5y &= 32 - 22 = 10;\\ \therefore\ y &= \frac{10}{5} = 2.\end{aligned}$$

$$\begin{aligned}\text{Also}\quad 2x + 3y + 4z &= 16,\\ \text{that is,}\quad 2x + 6 + 4 &= 16,\\ 2x &= 16 - 6 - 4 = 6;\\ \therefore\ x &= 3.\end{aligned}$$

The same method may be applied to any number of *independent* simple equations, in which the number of unknown quantities is the same as the number of equations.

Another method. $$\left.\begin{aligned}a_1x + b_1y + c_1z &= d_1{}^{*}\ \ldots\ldots(1)\\ a_2x + b_2y + c_2z &= d_2\ \ \ldots\ldots(2)\\ a_3x + b_3y + c_3z &= d_3\ \ \ldots\ldots(3)\end{aligned}\right\};\ \text{to find } x,\ y, \text{ and } z.$$

Multiply (2) by m, (3) by n, and to the resulting equations add (1); then we have

$$(a_1 + ma_2 + na_3)x + (b_1 + mb_2 + nb_3)y + (c_1 + mc_2 + nc_3)z = d_1 + md_2 + nd_3.$$

Now to find x, let the *arbitrary* multipliers m and n be such that the coefficients of y and z in this last equation are separately equal to 0; that is,

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[*] The small figures here give no particular *values* to the quantities to which they are annexed, a_1 and a_2 being as different as a and b; but it is often convenient to use the same letter thus slightly varied to mark some common meaning of such letters, and thereby assist the memory. Thus in this instance, a_1, a_2, a_3, have this common property, viz. that all are coefficients of x, a_1 in the 1st, a_2 in the 2nd, and a_3 in the 3rd, equation. Similarly for the coefficients of y and z.

$$\left.\begin{aligned} b_1 + mb_2 + nb_3 &= 0 \\ \text{and } c_1 + mc_2 + nc_3 &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} \text{or } b_1c_3 + mb_2c_3 + nb_3c_3 &= 0 \\ \text{and } b_3c_1 + mb_3c_2 + nb_3c_3 &= 0 \end{aligned}\right\},$$

$$\text{and } \therefore\ b_1c_3 - b_3c_1 + (b_2c_3 - b_3c_2)m = 0,$$

$$\text{or } m = \frac{b_3c_1 - b_1c_3}{b_2c_3 - b_3c_2};$$

$$\text{and similarly } n = \frac{b_1c_2 - b_2c_1}{b_2c_3 - b_3c_2}.$$

Then we have $x = \dfrac{d_1 + md_2 + nd_3}{a_1 + ma_2 + na_3}$, in which the above values may be substituted for m and n.

Similarly, by making the coefficients of x and z, or of x and y, separately equal to 0, the value of y, or of z, may be found.

Hence, the following Rule may be deduced, and will be found easy of application:—

To find x, multiply the 1st equation by $b_2c_3 - b_3c_2$, the 2nd by $b_3c_1 - b_1c_3$, and the 3rd by $b_1c_2 - b_2c_1$; then add together the resulting equations, and a simple equation will be obtained in which y and z do not appear.

A similar rule may be stated for finding either y or z; or having found the value of x, the equations are reduced to simple equations of two unknown quantities y and z, so that y and z may be found by any of the methods of Art. 196.

Ex. Given
$$\left.\begin{aligned} 2x + 3y + 4z &= 16 \ldots (1) \\ 3x + 2y - 5z &= 8 \ldots (2) \\ 5x - 6y + 3z &= 6 \ldots (3) \end{aligned}\right\}; \text{ required } x.$$

$$\begin{aligned} \text{Here } b_2c_3 - b_3c_2 &= 6 - 30 = -24, \\ b_3c_1 - b_1c_3 &= -24 - 9 = -33, \\ b_1c_2 - b_2c_1 &= -15 - 8 = -23; \end{aligned}$$

$$\left.\begin{aligned} \therefore \text{ from } (1) - 48x - 72y - 96z &= -384 \\ \ldots\ldots (2) - 99x - 66y + 165z &= -264 \\ \ldots\ldots (3) - 115x + 138y - 69z &= -138 \end{aligned}\right\};$$

$$\therefore \text{ adding } 262x = 786,$$

$$\text{or } x = \frac{786}{262} = 3.$$

198. That the unknown quantities may have definite values, there must be as many *independent* equations as unknown quantities. When there are *more* equations than unknown quantities, the value of any one of these quantities may be determined from different equations; and should the values thus found differ, the equations are incongruous; should they be the same, one or more of the equations are unnecessary. When there are *fewer* equations than unknown quantities, one of these quantities cannot be found, but in terms which involve some of the rest, whose values may be assumed at pleasure; and in such cases the number of answers is indefinite. Thus, if $x + y = a$, then $x = a - y$; and assuming y at pleasure, we obtain a value of x such, that $x + y = a$.

These equations must also be *independent*, that is, not deducible one from another.

Let $x + y = a$, and $2x + 2y = 2a$; these are not *independent* equations, since the latter equation being deducible from the former, it involves no different suppositions, nor requires any thing more for its truth, than that $x + y = a$ should be a just equation.

It is sometimes, however, not easy to discover at once whether proposed equations be *independent* or not. Thus in the equations

$$\left.\begin{aligned} x + 3y + 4z &= 9 \\ 3x - 2y + 17z &= 25 \\ x + 14y - z &= 11 \end{aligned}\right\}$$

it is not obvious at first sight that the third equation is derived from the other two. But by multiplying the first equation by 4, and subtracting the second, the result is the third equation; and accordingly the usual process, being applied to find x, y, z, would certainly fail.

As examples of *incongruous* equations, the following may be instanced: $x + y = 7$, and $3x + 3y = 30$, from which we get $7 = 10$; or, again $x + y = 7$, $3x - y = 1$, and $x + 2y = 10$, from which we get $10 = 12$.

PROBLEMS WHICH PRODUCE SIMPLE EQUATIONS.

199. From certain quantities which are known to investigate others which have a given relation to them, is the business of Algebra.

When a question is proposed to be resolved, we must first consider fully its meaning and conditions. Then substituting one or more of the symbols x, y, z &c. for such unknown quantities as appear most convenient, we must proceed as if they were already determined, and we wished to try whether they answer all the proposed conditions or not, till as many independent equations arise as we have assumed unknown quantities, which will always be the case if the question be properly limited (Art. 198); and by the solution of these equations the quantities sought will be determined.

Prob. 1. A bankrupt owes *A* twice as much as he owes *B*, and *C* as much as he owes *A* and *B* together; out of £300, which is to be divided amongst them, what must each receive?

Let x represent what *B* must receive, in pounds;

then $2x =$ what *A* must receive,

and $x + 2x$, or $3x =$ what *C* must receive;

amongst them they receive £300; therefore

$$x + 2x + 3x = 300,$$

$$6x = 300;$$

$$\therefore x = \frac{300}{6} = £50. \text{ what } B \text{ must receive.}$$

$$2x = £100. \text{ what } A \text{ must receive.}$$

$$3x = £150. \text{ what } C \text{ must receive.}$$

Prob. 2. To divide a line of 15 inches into two such parts, that one may be three fourths of the other.

Let $x =$ the number of inches in one part,

then $\frac{3x}{4} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$the other.

$$x + \frac{3x}{4} = 15, \text{ by the question,}$$

$$4x + 3x = 60,$$

$$7x = 60;$$

$$\therefore x = \frac{60}{7} = 8\tfrac{4}{7}, \text{ one part;}$$

$$\text{and } \frac{3x}{4} = \frac{3}{4} \times \frac{60}{7} = \frac{45}{7} = 6\tfrac{3}{7}, \text{ the other part.}$$

Prob. 3. If *A* can perform a piece of work in 8 days, and *B* the same in 10 days, in what time will they finish it together?

Let x be the time required, in days; and w the work.

In one day A performs $\frac{1}{8}$th part of the work, or $\frac{w}{8}$; therefore ı x days he performs $\frac{xw}{8}$. And in the same time B performs $\frac{xw}{10}$.

'herefore $\frac{xw}{8} + \frac{xw}{10} = w$, by the question,

$$\text{or } \frac{x}{8} + \frac{x}{10} = 1,$$

$$10x + 8x = 80,$$

$$18x = 80;$$

$$\therefore x = \frac{80}{18} = 4\frac{8}{18} = 4\frac{4}{9} \text{ days.}$$

Prob. 4. A workman was employed for 60 days, on condition ıat for every day he worked he should receive 15 pence, and for very day he played he should forfeit 5 pence; at the end of the ime he had 20 shillings to receive; required the number of days e worked.

Let x be the number of days he worked,

then $60 - x$ is the number he played,

$15x =$ his pay, in pence,

$0 - x) \times 5 = 300 - 5x =$ sum forfeited;

$15x - 300 + 5x = 240$, by the question,

$$20x = 240 + 300 = 540;$$

$\therefore x = 27$, the number of days he worked,

$60 - x = 33$, the number of days he played.

Prob. 5. How much rye, at four shillings and sixpence a ıshel, must be mixed with 50 bushels of wheat, at six shillings bushel, that the mixture may be worth five shillings a bushel?

Let x be the number of bushels required;

then $9x =$ the price of the rye in sixpences,

$600 =$ the price of the wheat in sixpences,

$(50+x)\ 10 =$ the price of the mixture ....

$\therefore 9x + 600 = 500 + 10x,$

$100 = x$, the number of bushels required.

PROB. 6. *A* and *B* engage together in play; in the first game *A* wins as much as he had and four shillings more, and finds he has twice as much as *B*; in the second game *B* wins half as much as he had at first and one shilling more, and then it appears that he has three times as much as *A*; what sum had each at first?

Let x be what *A* had, in shillings,

y what *B* had.

Then $2x + 4 =$ what *A* has after the first game;

$y - x - 4 =$ what *B* has;

$\therefore$ by the question $2x + 4 = 2y - 2x - 8,$

or $2y - 4x = 12,$

$y - 2x = 6.$

Also $y - x - 4 + \frac{y}{2} + 1 =$ what *B* has after the second game;

$2x + 4 - \frac{y}{2} - 1 =$ what *A* has;

$\therefore$ by the question $y - x - 4 + \frac{y}{2} + 1 = 6x + 12 - \frac{3y}{2} - 3,$

$$\text{or } y + \frac{y}{2} + \frac{3y}{2} - x - 6x = 12 - 3 + 4 - 1,$$

$$\left.\begin{aligned} \text{or } 3y - 7x &= 12 \\ \text{Also } y - 2x &= 6, \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore 3y - 6x &= 18, \\ \text{and } 3y - 7x &= 12, \end{aligned}\right\}$$

$$\therefore \text{ by subtraction } \quad x = 6, \text{ what } A \text{ had at first;}$$

$$\text{and} \quad y - 2x = 6, \text{ or } y - 12 = 6;$$

$$\therefore y = 18, \text{ what } B \text{ had.}$$

Prob. 7. A smuggler had a quantity of brandy which he expected would raise £9. 18*s.*; after he had sold 10 gallons, a revenue officer seized one third of the remainder, in consequence of which he makes only £8. 2*s.*; required the number of gallons he had, and the price per gallon.

Let x be the number of gallons;

then $\frac{198}{x}$ is the price per gallon, in shillings,

$\frac{x-10}{3}$ the quantity seized,

and $\frac{x-10}{3} \times \frac{198}{x}$ the value of the quantity seized, which appears by the question to be 36 shillings;

$$\therefore \frac{x-10}{3} \times \frac{198}{x} = 36,$$

$$(x - 10) \times 66 = 36x,$$

$$66x - 660 = 36x,$$

$$30x = 660;$$

$$\therefore x = 22, \text{ the number of gallons,}$$

and $\frac{198}{x} = \frac{198}{22} = 9$ shillings, the price per gallon.

Prob. 8. A and B play at bowls, and A bets B three shillings to two upon every game; after a certain number of games it appears that A has won three shillings; but had he ventured to bet five shillings to two, and lost one game more out of the same number, he would have lost thirty shillings: how many games did they play?

Let x be the number of games A won,

y the number B won,

then $2x$ is what A won of B, in shillings,

and $3y$ what B won of A.

$\therefore\ 2x - 3y = 3$, by the question.

Also $(x-1)\times 2$ is what A would win on the 2d supposition,

and $(y+1)\times 5$ what B would win ....

$$\therefore\ 5y + 5 - 2x + 2 = 30, \text{ by the question,}$$

$$\text{or}\quad 5y - 2x = 30 - 5 - 2 = 23;$$

$$\left.\begin{aligned} \therefore\quad 5y - 2x &= 23, \\ \text{and}\quad 2x - 3y &= 3; \end{aligned}\right\}$$

$$\text{by addition, } 5y - 3y = 26,$$

$$2y = 26;$$

$$\therefore\ y = 13.$$

$$\text{And}\quad 2x = 3 + 3y = 3 + 39 = 42;$$

$$\therefore\ x = 21;$$

and $x + y = 34$, the number of games required.

Prob. 9. A sum of money was divided equally amongst a certain number of persons; had there been three more, each would have received one shilling less, and had there been two fewer, each would have received one shilling more, than he did: required the number of persons, and what each received.

Let x be the number of persons,

y the sum each received, in shillings;

then xy is the sum divided,

$$\left.\begin{aligned} \text{and}\quad (x+3)\times(y-1) &= xy \\ \text{also}\quad (x-2)\times(y+1) &= xy \end{aligned}\right\} \text{by the question;}$$

$$\therefore\ xy - x + 3y - 3 = xy, \text{ or } -x + 3y = 3;$$

$$\text{and}\quad xy + x - 2y - 2 = xy, \text{ or }\quad x - 2y = 2;$$

$\therefore\ y = 5$ shillings, the sum received by each.

$$\text{And}\quad x - 2y = x - 10 = 2,$$

$\therefore\ x = 12$, the number of persons.

QUADRATIC EQUATIONS.

200. When the terms of an equation involve the square of the unknown quantity, but the first power does not appear, the value of the square is obtained by the preceding rules*; and by extracting the square root on both sides, the quantity itself is found.

Ex. 1. $5x^2 - 45 = 0$; to find x.

$$\text{By transp.}\quad 5x^2 = 45,$$
$$x^2 = 9;$$
$$\therefore \text{(Art. 191)},\ x = \sqrt{9} = \pm 3.$$

The signs + and − are both prefixed to the root, because the square root of a quantity may be either positive or negative (Art. 147). The sign of x may also be negative; but still x will be either equal to $+3$ or -3†.

Ex. 2. $ax^2 = bcd$; to find x.

$$x^2 = \frac{bcd}{a};$$
$$\therefore x = \pm\sqrt{\frac{bcd}{a}}.$$

201. If both the first and second powers of the unknown quantity be found in an equation, arrange the terms according to the dimensions of the unknown quantity, beginning with the highest, and transpose the known quantities to the other side; then, if the square of the unknown quantity be affected with a coefficient, divide all the terms by this coefficient, and if its sign be negative, change

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* It is obvious that the rules proved in Arts. 186—192, apply to *all* equations, *quadratic*, *cubic*, &c. as well as *simple*, because they are founded simply upon the *Axioms* (Arts. 79—82.)

† This may be shewn as follows:—suppose $x^2 = a^2$, then extracting the square root of both sides, since $\sqrt{x^2} = \pm x$, and $\sqrt{a^2} = \pm a$, we have

$$+x = +a \ldots\ldots (1),$$
$$+x = -a \ldots\ldots (2),$$
$$-x = +a \ldots\ldots (3),$$
$$-x = -a \ldots\ldots (4).$$

But it is evident that (1) and (4) are in fact the same equation, and also (2) and (3); so that $x = \pm a$ includes all the four equations.

the signs of all the terms (Art. 187), that the equation may be reduced to this form, $x^2 \pm px = \pm q$. Then add to both sides the square of half the coefficient of the first power of the unknown quantity, by which means the first side of the equation is made a complete square, (Art. 153), and the other consists of known quantities; and by extracting the square root of both sides, a simple equation is obtained, from which the value of the unknown quantity may be found.

Ex. 1. Let $x^2 + px = q$; now, we know that $x^2 + px + \frac{p^2}{4}$ is the square of $x + \frac{p}{2}$ (Art. 153); add therefore $\frac{p^2}{4}$ to both sides, and we have

$$x^2 + px + \frac{p^2}{4} = q + \frac{p^2}{4};$$

then by extracting the square root on both sides,

$$x + \frac{p}{2} = \pm \sqrt{q + \frac{p^2}{4}};$$ and by transposition,

$$x = -\frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

In the same manner, if $x^2 - px = q$,

$$x = \frac{p}{2} \pm \sqrt{q + \frac{p^2}{4}}.$$

Ex. 2. $x^2 - 12x + 35 = 0$; to find x.

By transposition $x^2 - 12x = -35$; and adding the square of $\frac{12}{2}$ or 6 to both sides of the equation,

$$x^2 - 12x + 36 = 36 - 35 = 1;$$

then extracting the square root of both sides,

$$x - 6 = \pm 1;$$

$\therefore$ $x = 6 \pm 1 = 7$ or 5; either of which, substituted for x in the original equation, answers the condition, that is, makes the whole equal to nothing.

Ex. 3. $$\frac{6}{x+1} + \frac{2}{x} = 3; \text{ to find } x.$$

$$6 + \frac{2x+2}{x} = 3x + 3$$

$$6x + 2x + 2 = 3x^2 + 3x$$

$$3x^2 - 5x = 2$$

$$x^2 - \frac{5x}{3} = \frac{2}{3}$$

$$x^2 - \frac{5x}{3} + \left(\frac{5}{6}\right)^2 = \frac{2}{3} + \frac{25}{36}$$

$$= \frac{24}{36} + \frac{25}{36} = \frac{49}{36}.$$

$$x - \frac{5}{6} = \pm \frac{7}{6};$$

$$\therefore\ x = \frac{5 \pm 7}{6}$$

$$= 2, \text{ or } -\frac{1}{3}.$$

202. Ex. 4. $x + \sqrt{5x+10} = 8$; to find x.

By transp. $\sqrt{5x+10} = 8 - x$,

squaring, $5x + 10 = 64 - 16x + x^2$,

$$x^2 - 21x = -54;$$

mpleting the square,

$$x^2 - 21x + \frac{441}{4} = \frac{441}{4} - 54,$$

$$= \frac{225}{4};$$

tracting the square root,

$$x - \frac{21}{2} = \pm \frac{15}{2};$$

$$\therefore\ x = \frac{21 \pm 15}{2} = 18, \text{ or } 3.$$

By this process two values of x are found; but on trial it appears, that 18 does not answer the conditions of the equation, if we suppose that $\sqrt{5x+10}$ represents the positive square root of $5x+10$. The reason is, that $5x+10$ is the square of $-\sqrt{5x+10}$ as well as of $+\sqrt{5x+10}$; thus by squaring both sides of the equation $\sqrt{5x+10}=8-x$, a new condition is introduced, and a new value of the unknown quantity corresponding to it, which had no place before. Here 18 is the value which corresponds to the supposition that

$$x-\sqrt{5x+10}=8.$$

It should be particularly observed, that since $+x\times+y$ is equal to $-x\times-y$, in the multiplication and involution of quantities new values are always introduced, which, if not again excluded by the nature of the question, will appear in the final equation.

203. If a quadratic equation appear under the form $ax^2\pm bx=\pm c$, the left hand side may be made a complete square, and the equation solved, by another method, as follows:—

Multiply the whole equation by $4a$, that is, four times the coefficient of x^2, then we have

$$4a^2x^2\pm 4abx=\pm 4ac;$$

add b^2, the square of the coefficient of x, then $4a^2x^2\pm 4abx+b^2=b^2\pm 4ac$

extract the square root, $2ax\pm b=\pm\sqrt{b^2\pm 4ac}$;

$$\therefore\ x=\frac{\pm\sqrt{b^2\pm 4ac}\mp b}{2a}.$$

Ex. 1. $\dfrac{6}{x+1}+\dfrac{2}{x}=3$; to find x.

$$6+\frac{2x+2}{x}=3x+3,$$

$$6x+2x+2=3x^2+3x,$$

$$3x^2-5x=2.$$

Multiplying by 4×3, or 12,

$$36x^2-60x=24;$$

adding 5^2, or 25, $36x^2-60x+25=24+25=49$;

$$\therefore\ 6x-5=\pm 7,$$

$$6x=5\pm 7=12,\text{ or }-2;$$

$$\therefore\ x=2,\text{ or }-\frac{1}{3}.$$

Ex. 2. $acx^2 - bcx + adx = bd$; to find x.

Here $acx^2 - (bc - ad)\,x = bd$;

multiply by $4ac$, $4a^2c^2x^2 - 4ac\,(bc - ad)\,x = 4abcd$,

add $(bc - ad)^2$,
$$4a^2c^2x^2 - 4ac\,(bc - ad)\,x + (bc - ad)^2 = (bc - ad)^2 + 4abcd = (bc + ad)^2.$$

xtract square root, $2acx - (bc - ad) = \pm (bc + ad)$;
$$2acx = bc - ad \pm (bc + ad) = 2bc, \text{ or } -2ad;$$
$$\therefore\ x = \frac{b}{a}, \text{ or } -\frac{d}{c}.$$

204. *A quadratic equation has no more than two distinct values of e unknown quantity which will satisfy it.*

For, if possible, let the equation $ax^2 + bx + c = 0$ have three disnct values of x, viz. α, β, γ.

$$\text{Then}\quad a\alpha^2 + b\alpha + c = 0 \ldots (1),$$
$$a\beta^2 + b\beta + c = 0 \ldots (2),$$
$$a\gamma^2 + b\gamma + c = 0 \ldots (3).$$

ubtracting (2) from (1), $a\,(\alpha^2 - \beta^2) + b\,(\alpha - \beta) = 0$;
$$\therefore\ a\,(\alpha + \beta) + b = 0 \ldots\ldots\ldots \text{(i.)}$$
$$\text{Again,}\quad a\,(\alpha^2 - \gamma^2) + b\,(\alpha - \gamma) = 0;$$
$$\therefore\ a\,(\alpha + \gamma) + b = 0 \ldots\ldots\ldots \text{(ii.)}$$

ubtracting (ii.) from (i.) $a\,(\beta - \gamma) = 0$.

But a is not equal to 0, for otherwise the proposed equation would ot be a quadratic equation;
$$\therefore\ \beta - \gamma = 0,$$
$$\text{or}\quad \beta = \gamma.$$

Hence a quadratic equation has not *three* distinct values of x, but it ay have two.

205. *In any quadratic equation of the form* $x^2 + px + q = 0$, $-p =$ *the m of the two values of* x, *and* q = *their product.*

Let α, β, be the two values of x, then
$$\alpha^2 + p\alpha + q = 0,$$
$$\text{and } \beta^2 + p\beta + q = 0;$$

$$\therefore\ \alpha^2-\beta^2+p(\alpha-\beta)=0;$$

$$\therefore\ \alpha+\beta+p=0,$$

$$\text{or } -p=\alpha+\beta\ \ldots\ldots(1).$$

$$\text{Again, } q=-p\alpha-\alpha^2$$

$$=(\alpha+\beta)\,\alpha-\alpha^2$$

$$=\alpha\beta\ldots\ldots\ldots\ldots(2).$$

COR. 1. If the equation be of the form $ax^2+bx+c=0$, then $x^2+\frac{b}{a}x+\frac{c}{a}=0$. Therefore, by what has been proved,

$$-\frac{b}{a}=\text{the sum of the two values of } x,$$

$$\text{and } \frac{c}{a}=\text{the product}\ldots\ldots\ldots\ldots\ldots\ldots$$

COR. 2. Hence also, if α, β be the *roots* of the equation $x^2+px+q=0$, it is proved that

$$x^2+px+q=x^2-(\alpha+\beta)\,x+\alpha\beta=0$$

$$=(x-\alpha)\,(x-\beta)=0;$$

from which it appears that if one 'root' of the equation be known, the other may be found by division, for

$$\frac{x^2+px+q}{x-\alpha}=x-\beta=0, \text{ which gives } x=\beta.$$

Again, conversely, if the roots of an equation be given, the equation can be found. For, if α, β be the roots, the equation must be

$$(x-\alpha)\,(x-\beta)=0,$$

$$\text{or } x^2-(\alpha+\beta)\,x+\alpha\beta=0.$$

Ex. Required the equation whose roots are 2 and 3.

$$\text{The equation is } (x-2)\,(x-3)=0,$$

$$\text{or}\quad x^2-5x+6=0.$$

206. The results proved in the last Art. shewing the relation between the values of x and the coefficients in the equation, are of use in several ways; first, in enabling us to *verify* the solution of any quadratic equation; secondly, in determining the values of the unknown quantities when an equation is proposed in which certain relations are already known respecting those values; and lastly, in solving various problems, reduced to quadratic equations, of which it is necessary for our purpose to know no more than the sum or the product of the values of the unknown quantity. Thus,

Ex. 1. $3x^2-5x=2$. The values of x (See Art. 201. Ex. 3.) are 2, and $-\frac{1}{3}$.

Now, without repeating the work, to see if these values are correct, we put the equation under the form $x^2-\frac{5}{3}x-\frac{2}{3}=0$, and since $2+\left(-\frac{1}{3}\right)=\frac{5}{3}$, and $2\times-\frac{1}{3}=-\frac{2}{3}$, we conclude, at once, that these and no other are the values required.

Ex. 2. $x^2-21x+54=0$; required the values of x, it being known that one of them is six times as great as the other.

Suppose a to represent one of the values,

then $6a$ is the other,

and their sum, or $7a=21$,

$\therefore\ a=3$,

and the values required are 3 and 18.

Ex. 3. If α and β be the values of x in the equation $ax^2+bx+c=0$, find the value of $\frac{1}{\alpha}+\frac{1}{\beta}$.

Dividing the equation by x^2, $a+\frac{b}{x}+\frac{c}{x^2}=0$,

$$\text{or } \frac{c}{x^2}+\frac{b}{x}+a=0. \quad \text{Assume } y=\frac{1}{x},$$

$$\text{then } cy^2+by+a=0,$$

$$\text{or } y^2+\frac{b}{c}y+\frac{a}{c}=0;$$

and the values of y are $\frac{1}{\alpha}$, $\frac{1}{\beta}$, since $y=\frac{1}{x}$,

$$\therefore\ \frac{1}{\alpha}+\frac{1}{\beta}=-\frac{b}{c}. \quad \text{(Art. 205.)}$$

207. Since every quadratic equation may be reduced to the form $x^2+px+q=0$, in which p and q may be positive or negative, we assume this as the *general* equation including every other.

$$\text{Then, since } x^2+px=-q,$$

$$x^2+px+\frac{p^2}{4}=\frac{p^2-4q}{4},$$

$$x+\frac{p}{2}=\pm\frac{1}{2}\sqrt{p^2-4q},$$

$$\text{and } x=-\frac{p}{2}\pm\frac{1}{2}\sqrt{p^2-4q},$$

which are the only two values of x that will satisfy the equation.

Now, from this result, it follows,

1st. That there is no possible value of x, if $p^2 < 4q$.

2dly. That the values of x are equal to each other, and each equal to $-\frac{p}{2}$, if $p^2 = 4q$.

3rdly. That there are two unequal values of x, whose sum is $-p$, if $p^2 > 4q$.

Again, it appears that

1st. If q be negative, since $\sqrt{p^2 - 4q}$ will then be greater than p and always possible, there can be but one *positive* value of x.

2dly. If p be negative, q positive, and $p^2 > 4q$, there will be two *positive* values of x.

The last two conclusions may also be deduced from Art. 205.

Cor. Similar conclusions may be drawn with regard to the equation $ax^2 + bx + c = 0$, by substituting $\frac{b}{a}$ for p, and $\frac{c}{a}$ for q. Thus, if a proposed equation be of the form $bx - ax^2 = c$, in which a, b, c are positive quantities, there will be two *positive* values of x, when $4ac < b^2$; and no possible value of x when $4ac > b^2$.

208. If an equation appear under the form

$$(x + a)X = 0,$$

in which X represents an expression involving x, the unknown quantity; it is evident that either $x + a = 0$, or $X = 0$, that is, $x = -a$ is one solution of the equation, as well as those which are found by proceeding with $X = 0$. So that, *whenever an equation is simplified by division, or the omission of a factor, if the divisor or factor contain the unknown quantity, one solution at least of the equation will be found by putting that divisor or factor equal to 0.*

Thus, let $x^2 + 3x = 7x$; the whole equation is divisible by x, therefore $x = 0$ is one solution.

Again, let $x^2 - 5x + 6 = 0$. This may be put under the form

$$(x - 2)(x - 3) = 0;$$

$$\left.\begin{array}{l} \therefore\ x - 2 = 0, \text{ or } x = 2 \\ \text{and } x - 3 = 0, \text{ or } x = 3 \end{array}\right\}; \text{ which are the only values of } x.$$

By this method, therefore, the necessity for solving a quadratic in the usual way may sometimes be superseded.

Ex. Given $(x - c)\sqrt{ab} - (a - b)\sqrt{cx} = 0$; required x.

$$\text{Here } x\sqrt{ab} - c\sqrt{ab} - a\sqrt{cx} + b\sqrt{cx} = 0;$$

$$\therefore\ \sqrt{bx}(\sqrt{ax} + \sqrt{bc}) - \sqrt{ac}(\sqrt{ax} + \sqrt{bc}) = 0,$$

$$\text{or } (\sqrt{bx} - \sqrt{ac})(\sqrt{ax} + \sqrt{bc}) = 0;$$

$$\therefore \sqrt{bx} - \sqrt{ac} = 0, \text{ and } \sqrt{ax} + \sqrt{bc} = 0;$$

$$\therefore bx = ac, \text{ and } ax = bc,$$

$$x = \frac{ac}{b}, \text{ and } x = \frac{bc}{a}.$$

209. Every equation where the unknown quantity is found in two terms, and its index in one is twice as great as in the other, may be resolved in the same manner as a quadratic.

Ex. 1. $x + 4x^{\frac{1}{2}} = 21$; required x.

$$x + 4x^{\frac{1}{2}} + 4 = 21 + 4 = 25,$$

$$x^{\frac{1}{2}} + 2 = \pm 5,$$

$$x^{\frac{1}{2}} = \pm 5 - 2 = 3, \text{ or } -7;$$

$$\therefore x = 9, \text{ or } 49.$$

Ex. 2. $x^{-1} + x^{-\frac{1}{2}} = 6$; required x.

$$x^{-1} + x^{-\frac{1}{2}} + \frac{1}{4} = 6 + \frac{1}{4} = \frac{25}{4},$$

$$x^{-\frac{1}{2}} + \frac{1}{2} = \pm \frac{5}{2},$$

$$x^{-\frac{1}{2}} = \frac{-1 \pm 5}{2} = 2, \text{ or } -3,$$

$$x^{\frac{1}{2}} = \frac{1}{2}, \text{ or } -\frac{1}{3};$$

$$\therefore x = \frac{1}{4}, \text{ or } \frac{1}{9}.$$

Ex. 3. $y^4 - 6y^2 - 27 = 0$; required y.

$$y^4 - 6y^2 = 27,$$

$$y^4 - 6y^2 + 9 = 27 + 9 = 36,$$

$$y^2 - 3 = \pm 6,$$

$$y^2 = 3 \pm 6 = 9, \text{ or } -3;$$

$$\therefore y = \pm 3, \text{ or } \pm \sqrt{-3}.$$

Ex. 4. $y^6 + ry^3 + \frac{q^3}{27} = 0$; required x.

$$y^6 + ry^3 = -\frac{q^3}{27},$$

$$y^6 + ry^3 + \frac{r^2}{4} = \frac{r^2}{4} - \frac{q^3}{27},$$

$$y^6 + \frac{r}{2} = \pm\sqrt{\frac{r^2}{4} - \frac{q^3}{27}},$$

$$y^3 = -\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}};$$

$$\therefore\ y = \sqrt[3]{-\frac{r}{2} \pm \sqrt{\frac{r^2}{4} - \frac{q^3}{27}}}.$$

210. Some other equations may be conveniently solved as quadratics, that is, by completing the square, when they can be made to assume the form

$$X^2 + pX = q,$$

X representing a compound expression involving the unknown quantity.

Ex. 1. $ax^2 + \sqrt{ax^2 - bx + c} = bx$; required x.

By transp. $ax^2 - bx + \sqrt{ax^2 - bx + c} = 0,$

adding c, $ax^2 - bx + c + \sqrt{ax^2 - bx + c} = c,$

completing the square,

$$(ax^2 - bx + c) + \sqrt{ax^2 - bx + c} + \frac{1}{4} = c + \frac{1}{4}$$

$$\sqrt{ax^2 - bx + c} + \frac{1}{2} = \pm\sqrt{c + \frac{1}{4}}$$

$$\sqrt{ax^2 - bx + c} = \frac{\pm\sqrt{4c+1} - 1}{2}$$

$$ax^2 - bx + c = \left\{\frac{\pm\sqrt{4c+1} - 1}{2}\right\}^2;$$

the equation is thus reduced to a common quadratic, from which x may be found by the usual method.

Ex. 2. $x^2 - x + 5\sqrt{2x^2 - 5x + 6} = \frac{1}{2}(3x + 33)$; required x.

Here $2x^2 - 2x + 10\sqrt{2x^2 - 5x + 6} = 3x + 33,$

$$2x^2-5x+6+10\sqrt{2x^2-5x+6}=39,$$

$$(2x^2-5x+6)+10\sqrt{2x^2-5x+6}+25=25+39=64,$$

$$\sqrt{2x^2-5x+6}+5=\pm 8,$$

$$\sqrt{2x^2-5x+6}=\pm 8-5=3, \text{ or } -13,$$

$$2x^2-5x+6=9, \text{ or } 169;$$

hich leaves two common quadratics for solution.

211. When there are more equations and unknown quantities ian one, a single equation involving only one of the unknown uantities may sometimes be obtained by the rules laid down for ie solution of simple equations; and one of the unknown quanti-es being discovered, the others may be obtained by substituting s value in the other equations.

Ex. $$\left.\begin{aligned} x-\frac{x-y}{2}&=4 \\ y-\frac{x+3y}{x+2}&=1 \end{aligned}\right\}; \text{ to find } x \text{ and } y.$$

From the first equation, $2x-x+y=8,$

$$x+y=8,$$

$$x=8-y.$$

From the 2nd equation, $xy+2y-x-3y=x+2,$

or $xy-2x-y=2,$

by substitution, $(8-y)y-2(8-y)-y=2,$

$$8y-y^2-16+2y-y=2,$$

$$9y-y^2=16+2=18,$$

$$y^2-9y=-18,$$

$$y^2-9y+\frac{81}{4}=\frac{81}{4}-18=\frac{9}{4},$$

$$y-\frac{9}{2}=\pm\frac{3}{2};$$

$$\therefore\ y=\frac{9\pm 3}{2}=6, \text{ or } 3.$$

And $x=8-y=2$, or 5.

The solution of equations will often be rendered more simple by particular artifices, the proper application of which is best learned by experience*.

Ex. 1. $$\left.\begin{aligned} x^2+y^2&=65 \\ xy&=28 \end{aligned}\right\};\text{ to find } x \text{ and } y.$$

From the second equation, $2xy = 56$,

adding this to the 1st, $\left.\begin{aligned} x^2+2xy+y^2 &= 121 \\ \text{subtracting it, } x^2-2xy+y^2 &= 9 \end{aligned}\right\}$

extracting the square roots, $\left.\begin{aligned} x+y &= \pm 11 \\ \text{and } x-y &= \pm 3 \end{aligned}\right\}.$

$$\left.\begin{aligned} \text{adding, } 2x &= \pm 14 \\ \text{subtracting, } 2y &= \pm 8 \end{aligned}\right\}$$

$$\therefore x = 7, \text{ or } -7,$$

$$\text{and } y = 4, \text{ or } -4.$$

Ex. 2. $$\left.\begin{aligned} \frac{x^2}{y}+\frac{y^2}{x} &= 18 \\ x+y &= 12 \end{aligned}\right\};\text{ to find } x \text{ and } y.$$

$$\left.\begin{aligned} \text{From first equation, } x^3+y^3 &= 18xy \\ \text{from second, } x^3+y^3+3xy(x+y) &= 1728 \end{aligned}\right\}$$

by substitution, $18xy+36xy = 1728$,

$$54xy = 1728;$$

$$\therefore xy = 32.$$

$$\left.\begin{aligned} \text{Also } x^2+2xy+y^2 &= 144 \\ \text{and } 4xy &= 128 \end{aligned}\right\}$$

$$\therefore x^2-2xy+y^2 = 16,$$

$$\left.\begin{aligned} x-y &= \pm 4 \\ \text{and } x+y &= 12 \end{aligned}\right\}$$

$$\left.\begin{aligned} \therefore 2x &= 16, \text{ or } 8, \\ \text{and } 2y &= 8, \text{ or } 16 \end{aligned}\right\}$$

$$\therefore x = 8, \text{ or } 4,$$

$$\text{and } y = 4, \text{ or } 8.$$

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* Many of these artifices are pointed out in the Appendix.

212. It may sometimes be of use to substitute for one of the nown quantities the product of the other and a third unknown ntity.

This substitution may be successfully applied whenever the sum he *dimensions* (Art. 63) of the *unknown quantities* in every m of each equation is the same.

Ex. $\left.\begin{aligned} x^2 + xy &= 12 \\ xy - 2y^2 &= 1 \end{aligned}\right\}$; to find x and y.

Let $vy = x$,

$$\left.\begin{aligned} \text{then}\quad v^2y^2 + vy^2 &= 12 \\ \text{and}\quad vy^2 - 2y^2 &= 1 \end{aligned}\right\}$$

$$\text{from the former}\quad y^2 = \frac{12}{v^2 + v},$$

$$\text{from the latter}\quad y^2 = \frac{1}{v - 2};$$

$$\therefore \frac{12}{v^2 + v} = \frac{1}{v - 2},$$

$$\text{or } v^2 + v = 12v - 24,$$

$$v^2 - 11v = -24,$$

$$v^2 - 11v + \frac{121}{4} = \frac{121}{4} - 24 = \frac{25}{4},$$

$$v - \frac{11}{2} = \pm\frac{5}{2};$$

$$\therefore v = \frac{11 \pm 5}{2} = 8, \text{ or } 3.$$

$$\text{And}\quad y^2 = \frac{1}{v - 2} = \frac{1}{6}, \text{ or } 1;$$

$$\therefore y = \pm\frac{1}{\sqrt{6}}, \text{ or } \pm 1.$$

$$\text{And}\quad x = vy = \pm\frac{8}{\sqrt{6}}, \text{ or } \pm 3.$$

213. The operation may sometimes be facilitated by substituting for the unknown quantities the sum and difference of two others

This artifice may be used, when the unknown quantities in each equation are similarly involved.

Ex. $\left.\begin{array}{l} x+y=4, \\ (x^2+y^2)(x^3+y^3)=280 \end{array}\right\}$; to find x and y.

$$\begin{aligned} \text{Assume } x &= z+v, \\ \text{and } y &= z-v, \\ \text{then } x+y &= 2z = 4; \\ \therefore\ z &= 2. \\ \text{Also } x^2+y^2 &= (2+v)^2+(2-v)^2, \\ &= 8+2v^2, \\ \text{and } x^3+y^3 &= (2+v)^3+(2-v)^3, \\ &= 8+12v+6v^2+v^3+8-12v+6v^2-v^3, \\ &= 16+12v^2; \end{aligned}$$

$$\begin{aligned} \therefore\ (8+2v^2)(16+12v^2) &= 280, \\ \text{or } (4+v^2)(4+3v^2) &= 35, \\ 16+16v^2+3v^4 &= 35, \\ v^4+\frac{16}{3}v^2 &= \frac{19}{3}, \\ v^4+\frac{16}{3}v^2+\frac{64}{9} &= \frac{19}{3}+\frac{64}{9} = \frac{121}{9}, \\ v^2+\frac{8}{3} &= \pm\frac{11}{3}; \\ \therefore\ v^2 &= \frac{\pm 11-8}{3} = 1, \text{ or } -\frac{19}{3}, \\ \text{and } v &= 1; \\ \therefore\ x &= z+v = 3, \\ \text{and } y &= z-v = 1. \end{aligned}$$

OBS. In algebraical analysis it is frequently useful to observe whether the algebraical expressions under consideration are *homogeneous* or not, that is, whether the 'dimensions' of every term be the same or not; for, if this homogeneity be found at first, no legitimate operation can destroy it; or, if it be not found at first, it cannot be introduced; and thus an easy test is afforded, to a certain extent, of the accuracy of each succeeding step in the analysis.

For example, if the equation

$$ax^2 + b^2x + c^3 = 0,$$

e proposed for solution, in which every term is of three dimensions, that , which is homogeneous, every step in the process will present an homoeneous equation, if it be correct.

As a simple case it may be well to observe that, if the proposed uation be *homogeneous*, the final result must be so. A proper attenon to this observation will frequently detect an error in the process of lving an equation.

PROBLEMS PRODUCING QUADRATIC EQUATIONS.

214. Prob. I. A person bought a certain number of oxen r 80 guineas, and if he had bought 4 more for the same sum, hey would have cost a guinea a piece less; required the number f oxen and price of each.

Let x be the number of oxen,

then $\frac{80}{x}$ is the price of each, in guineas,

and $\frac{80}{x+4}$ the price of each on the second supposition;

$$\therefore \frac{80}{x+4} = \frac{80}{x} - 1, \text{ by the question,}$$

$$80 = \frac{80x + 320}{x} - x - 4,$$

$$80x = 80x + 320 - x^2 - 4x,$$

$$x^2 + 4x = 320,$$

$$x^2 + 4x + 4 = 324,$$

$$x + 2 = \pm 18;$$

$$\therefore x = \pm 18 - 2 = 16, \text{ or } -20, \text{ numb. of oxen};$$

$$\text{and } \frac{80}{x} = \frac{80}{16} = 5 \text{ guineas, the price of each.}$$

In this, and in many other cases, especially in the solution f philosophical questions, we deduce from the algebraical process nswers which do not correspond with the conditions. The reason eems to be, that the algebraical expression is more general than e common language; and the equation, which is a proper repre- ntation of the conditions, will also express other conditions, and answer other suppositions. In the foregoing instance x may either represent a positive or a negative quantity, and cannot in the operation represent a positive quantity alone (Art. 202); and the equation

$$\frac{80}{x+4} = \frac{80}{x} - 1,$$

when x is negative, or represents the diminution of stock, will be a proper expression for the solution of the following problem: A person *sells* a certain number of oxen for 80 guineas; and had he sold 4 fewer for the same sum, he would have received a guinea a piece more for them; required the number sold.

215. **Prob. II.** To divide a line of 20 inches into two such parts, that the rectangle under the whole and one part may be equal to the square of the other part.

Let x be the greater part, then will $20 - x$ be the less,

and $x^2 = (20 - x) \times 20 = 400 - 20x$, by the question,

$$x^2 + 20x = 400,$$
$$x^2 + 20x + 100 = 400 + 100 = 500,$$
$$x + 10 = \pm\sqrt{500};$$
$$\therefore\ x = \sqrt{500} - 10, \text{ or } -\sqrt{500} - 10.$$

The observation contained in the preceding article may be applied here; and it is to be remarked, that the negative values thus deduced are not insignificant, or useless. Here the negative value shews, that if the line be produced $\sqrt{500} + 10$ inches, the square of the part produced is equal to the rectangle under the line given and the line made up of the whole and part produced.

216. **Prob. III.** To find two numbers, whose sum, product, and the sum of whose squares, are equal to each other.

Let $x + y$ and $x - y$ be the numbers,

their sum is $2x$,

their product $x^2 - y^2$,

the sum of their squares $2x^2 + 2y^2$,

and by the question $2x = 2x^2 + 2y^2$,

or $x = x^2 + y^2$.

Also $2x = x^2 - y^2$,

adding $3x = 2x^2$;

$$\therefore\ x = \frac{3}{2}.$$

$$2x = x^2 - y^2,$$

or $3 = \frac{9}{4} - y^2$,

$$y^2 = \frac{9}{4} - 3 = \frac{9 - 12}{4} = \frac{-3}{4};$$

$$\therefore\ y = \pm \frac{\sqrt{-3}}{2}.$$

Hence $x + y = \frac{3 + \sqrt{-3}}{2}$,

and $x - y = \frac{3 - \sqrt{-3}}{2}$;

oth of which are "impossible" quantities (Art. 183), a conclusion hich shews that there are no such numbers as the question supposes.

A collection of Problems with their Solutions will be found in the ppendix.

SCHOLIUM.

By the method of solution pursued in Art. 202 it is clear that *both* e resulting values of the unknown quantity may be those of a different uation and not of the proposed one; for if the proposed equation be of e form

$$ax + \sqrt{bx + c} = d,$$

e solution effected may be that of the equation

$$ax - \sqrt{bx + c} = d,$$

d it is impossible to say, without trial, to which equation either of the sulting values of x belongs.

That there is no value of x which will satisfy *both* equations is easily roved. For, if possible, let there be such value; then, for that value,

$$\sqrt{bx + c} = 0, \qquad \text{or } x = -\frac{c}{b},$$

value of x which will satisfy *neither* equation.

Hence it appears, that after solving an equation of the above form by the usual method, it still remains doubtful whether *either* of the values of the unknown quantity obtained will satisfy the equation; and if one of the two be the value sought, it remains doubtful which it is.

Thus from the equation

$$3x + \sqrt{30x - 71} = 5,$$

the values of x obtained are 4, and $2\frac{8}{9}$, *neither* of which will satisfy the equation.

And from the equation

$$3x + \sqrt{2x - 2} = 7,$$

the values of x obtained are 3, and $1\frac{8}{9}$, of which only the fractional, and not the integral, value will satisfy the equation.

The fact is, that in the former instance both values of x are the values belonging to the equation

$$3x - \sqrt{30x - 71} = 5;$$

in the latter $x = 3$ belongs to the equation

$$3x - \sqrt{2x - 2} = 7,$$

and the other solution $x = 1\frac{8}{9}$ to the equation as proposed.

Whether there be any values of x or not which will satisfy the equation $3x + \sqrt{30x - 71} = 5$ we cannot say: all that we know is, that the common method of solution will not produce them.

Where then, it may be asked, lies the fallacy in the process whereby we obtain a false result? It is here. We assume, as an axiom, that if the same root of equal quantities be extracted, those roots are equal to each other in all cases; whereas we know, that they *may* be unequal. For instance, retracing the steps in the following operation,

$$ax - \sqrt{bx + c} = d,$$

$$ax - d = \sqrt{bx + c},$$

$$(ax - d)^2 = bx + c,$$

we assume, that the same value of x which satisfies the last of these equations must also satisfy the preceding one; but this may not be the case since it may be the value which satisfies

$$-(ax - d) = \sqrt{bx + c},$$

or $$ax + \sqrt{bx + c} = d,$$

instead of $$ax - \sqrt{bx + c} = d,$$

the equation we commonly assume to be satisfied by that value, when it is the proposed equation, whose solution is required.

The fact is, that the equation really solved is not the proposed one

$$ax - \sqrt{bx + c} = d,$$

but $(ax + \sqrt{bx + c})\,(ax - \sqrt{bx + c}) = d^2$;

and it is quite a chance whether both or either of the values of x obtained belong to the proposed equation.

It is certain, however, that the values obtained belong to one or other of the two distinct equations

$$ax + \sqrt{bx + c} = d,$$

$$ax - \sqrt{bx + c} = d.$$

From what has been said it appears, that the results which are commonly obtained as solutions of quadratic equations, *when those equations are given in an irrational form*, require to be verified, before they can be depended upon.

Hence also, *when a proposed problem depends upon the solution of an irrational quadratic equation, the problem may, or may not, have those solutions which appear as solutions of the quadratic equation.* This conclusion, it must be evident, is of too important a nature, to be safely overlooked.

INEQUALITIES.

If one quantity be greater or less than another, or than nothing, and this be expressed algebraically, it is called an *Inequality.*

Thus, $x - a > b - x$ is an *Inequality*, of which $x - a$ forms one *side*, and $b - x$ the other.

217. *Any quantity may be added to, or subtracted from, each side of an inequality, and the sign of inequality remain as before.*

Thus, if $a > b$, $a \pm x > b \pm x$; for if $a > b$, it is evident that $a + x > b + x$. Similarly, if $a < b$, $a + x < b + x$.

Again, if $a > b$, it is evident that $a - x > b - x$, as long as x is not greater than a or b. If $x > b$, but not greater than a, then it is evident that $a - x > b - x$, for a positive quantity must be greater than a negative one. If $x > a$ and $> b$, then both sides $a - x$, and $b - x$, are negative; but a is nearer to x than b is to x, therefore $a - x < b - x$ independently of the signs; and of two negative quantities that is the greater, which is the smaller when the signs are omitted; therefore, in this case also, $a - x > b - x$.

Similarly, if $a < b$, in all cases $a - x < b - x$.

Cor. Hence any quantity may be transposed (as in equations) from one side of an inequality to the other by changing its sign. Thus, if

$$a^2 + b^2 > 2ab + c^2,$$

$$a^2 + b^2 - 2ab > 2ab - 2ab + c^2,$$

$$\text{or } (a - b)^2 > c^2.$$

218. If $a > b$, and $c > d$, and $e > f$, &c., then it is evident that

$$a + c + e + \&c. > b + d + f + \&c.;$$

But if $a > b$, and $c > d$, it does not always follow that $a - c > b - d$, for a may be more nearly equal to c, than b is to d. Thus $9 > 6$, and $7 > 2$, but $9 - 7$, or 2, is not greater than $6 - 2$, or 4.

219. *If every term on each side of an inequality be multiplied or divided by any positive quantity, the sign of inequality will remain as before.*

Thus, if $a > b$, $2a > 2b$, $6a > 6b$, &c.; or, if $-a > -b$, $-2a > -2b$, $-6a > -6b$; &c. as is sufficiently manifest.

Cor. Hence an inequality may be cleared of fractions by multiplying both sides by the product of the denominators of all the fractions, or by the least common multiple of them all; provided the multiplier is a *positive* quantity.

Ex. If $\frac{a}{b^2} + \frac{b}{a^2} > \frac{1}{a} + \frac{1}{b}$, multiplying by $a^2 b^2$,

$$a^3 + b^3 > ab^2 + a^2 b.$$

But, if all the terms of an inequality be multiplied or divided by a *negative* quantity, the sign of inequality is reversed, that is, $>$ is changed into $<$, or $<$ into $>$.

Thus, for example, $6 > 4$, but 6×-2 or $-12 < 4 \times -2$ or -8. Also $\frac{6}{-2}$ or $-3 < \frac{4}{-2}$ or -2.

Hence, if we multiply or divide the terms of an inequality by any *algebraical* quantity, it will be requisite to know whether the quantity is positive or negative.

Cor. Also, if the signs of all the terms of an inequality be changed, the sign of inequality is reversed, for this is equivalent to multiplying each side by -1.

220. *Both sides of an inequality may be raised to any power or any root of them extracted, and the sign of inequality remain as before, provided that each side is a positive quantity.*

Thus, $7 > 5$ and 7^2 or $49 > 5^2$ or 25, 7^3 or $343 > 5^3$ or 125, and so on.

But, if either side be negative, then no general conclusion can be stated as to the resulting inequality. For $-3 < 4$, and $(-3)^2$ or $9 < 4^2$ or 16. Also $-2 > -3$; but $(-2)^2$ or $4 < (-3)^2$ or 9.

Similarly $16 < 25$, and $\sqrt{16}$ or $4 < \sqrt{25}$ or 5. But, if either or both the sides be negative, no conclusion can be drawn as to their square roots.

Hence, if the sides of an *algebraical* inequality be raised to any power, or any root of them be extracted, we must first know whether the sides of the proposed inequality are positive or negative: otherwise the conclusion may not be correct in all cases.

221. *If the same quantity or two equal quantities be divided by each side of an inequality, the inequality will be reversed.*

For $3 < 6$, but $\frac{10}{3} > \frac{10}{6}$; and so also in any similar case.

Hence it appears, from this and the preceding articles, that the rules which belong to equations must not be inconsiderately applied to inequalities, since these latter have distinct rules of their own materially differing from those of the former.

Ex. 1. Shew that $a^2 + b^2 + c^2 > ab + ac + bc$, unless $a = b = c$.

Since every quantity, when squared, is positive, and therefore greater than 0, we have

$$(a - b)^2 > 0, \text{ unless } a = b,$$

$$\text{or } a^2 - 2ab + b^2 > 0,$$

$$\text{or } a^2 + b^2 > 2ab, \quad \text{(Art. 217. Cor.)}$$

$$\text{Similarly, } a^2 + c^2 > 2ac, \text{ unless } a = c,$$

$$\text{and } b^2 + c^2 > 2bc, \text{ unless } b = c;$$

$$\therefore \text{ by addition } 2a^2 + 2b^2 + 2c^2 > 2ab + 2ac + 2bc, \quad \text{(Art. 218.)}$$

$$\text{or } a^2 + b^2 + c^2 > ab + ac + bc. \quad \text{(Art. 219.)}$$

Ex. 2. Shew that $x^5 + y^5 > x^4y + y^4x$.

$$\text{Here } x^5 + y^5 > x^4y + y^4x,$$

$$\text{if } x^5 - x^4y > y^4x - y^5, \text{ (Art. 217)},$$

$$\text{if } x^4(x - y) > y^4(x - y), \text{ (1)}$$

$$\text{if } x^4 > y^4, \text{ (Art. 219, supposing } x > y),$$

$$\text{if } x > y, \text{ (Art. 220), which it is by supposition.}$$

Or, supposing $x < y$, then from (1) we have

$$\text{if } x^4 < y^4 \text{ (Art. 219)},$$

if $x < y$ (Art. 220), which again is so by supposition.

Hence, whether $x >$ or $< y$,

$$x^5 + y^5 > x^4y + y^4x.$$

Ex. 3. Given that $\frac{x+2}{4} + \frac{x}{3} < \frac{x-4}{2} + 3$, and $> \frac{x+1}{2} + \frac{1}{3}$, find x.

$$\frac{x+2}{4} + \frac{x}{3} < \frac{x-4}{2} + 3, \text{ and } > \frac{x+1}{2} + \frac{1}{3};$$

$\therefore$ mult. by 12, $3x + 6 + 4x < 6x - 24 + 36$, and $> 6x + 6 + 4$, (Art. 219.)

$$\text{or } 7x+6<6x+12, \text{ and } >6x+10,$$

$$x<6, \text{ and } >4, \text{ (Art. 217).}$$

Hence x is any number between 4 and 6; and, if it be a whole number, $x=5$.

Ex. 4. Which is greater $\sqrt{10}+\sqrt{7}$, or $\sqrt{19}+\sqrt{3}$?

$$\sqrt{10}+\sqrt{7} > \text{ or } < \sqrt{19}+\sqrt{3};$$

according as $17+2\sqrt{70} > \text{ or } < 22+2\sqrt{57}$, squaring by Art. 220,

$$\text{............} \quad 2\sqrt{70} > \text{ or } < 5+2\sqrt{57}, \text{ (Art. 217),}$$

$$\text{............} \quad 280 > \text{ or } < 253+20\sqrt{57}, \text{ (Art. 220),}$$

$$\text{............} \quad 27 > \text{ or } < 20\sqrt{57}.$$

Now 27 is clearly less than $20\sqrt{57}$, therefore $\sqrt{19}+\sqrt{3}$ is the greater of the two proposed quantities.

Ex. 5. Shew that $\dfrac{a+b+c+d}{p+q+r+s}>$ the least and $<$ the greatest of the fractions $\dfrac{a}{p}, \dfrac{b}{q}, \dfrac{c}{r}, \dfrac{d}{s}$, each letter representing a positive quantity.

Let G be a quantity greater than, and g a quantity less than, any of the fractions $\dfrac{a}{p}, \dfrac{b}{q}, \dfrac{c}{r}, \dfrac{d}{s}$. Then

$$\frac{a}{p}<G, \quad \frac{b}{q}<G, \quad \frac{c}{r}<G, \quad \frac{d}{s}<G,$$

$$\frac{a}{p}>g, \quad \frac{b}{q}>g, \quad \frac{c}{r}>g, \quad \frac{d}{s}>g.$$

$$\left.\begin{aligned} \therefore\ a<pG,\ b<qG,\ c<rG,\ d<sG,\\ a>pg,\ \ b>qg,\ \ c>rg,\ \ d>sg,\end{aligned}\right\}\text{ Art. 219.}$$

$$\left.\begin{aligned} \therefore\ a+b+c+d<(p+q+r+s)\,G,\\ \text{and } a+b+c+d>(p+q+r+s)\,g,\end{aligned}\right\}\text{ Art. 218.}$$

$$\therefore\ \frac{a+b+c+d}{p+q+r+s}<G \text{ and } >g. \text{ (Art. 219.)} \qquad \text{Q. E. D.}$$

RATIOS.

222. Ratio is the relation which one quantity bears to another in respect of magnitude, the comparison being made by considering what multiple, part, or parts, one is of the other.

Thus, in comparing 6 with 3, we observe that it has a certain magnitude with respect to 3, which it contains twice; again, in comparing it with 2, we see that it has a different relative magnitude, for it contains 2 three times, or it is greater when compared with 2 than it is when compared with 3. The ratio of a to b is usually expressed by two points placed between them, thus, $a : b$; and the former, a, is called the *antecedent* of the ratio, the latter, b, the *consequent.*

223. COR. 1. When one antecedent is the same multiple, part, or parts, of its consequent, that another antecedent is of its consequent, the ratios are equal. Thus, the ratio of $4 : 6$ is equal to the ratio of $2 : 3$, that is, 4 has the same magnitude when compared with 6, that 2 has when compared with 3, since $\frac{4}{6} = \frac{2}{3}$. The ratio of $a : b$ is equal to the ratio of $c : d$, if $\frac{a}{b} = \frac{c}{d}$, because $\frac{a}{b}$ and $\frac{c}{d}$ represent the multiple, part, or parts, that a is of b, and c of d.

Hence it appears, that the *measure* of any ratio $a : b$ is the fraction $\frac{a}{b}$.

224. COR. 2. If the terms of a ratio be multiplied or divided by the same quantity, the ratio is not altered.

For $\frac{a}{b} = \frac{ma}{mb}$ (Art. 101); $\therefore a : b = ma : mb$.

225. COR. 3. That ratio is greater than another, whose antecedent is the greater multiple, part, or parts, of its consequent. Thus the ratio $7 : 4$ is greater than the ratio $8 : 5$; because $\frac{7}{4}$, or $\frac{35}{20}$, is greater than $\frac{8}{5}$, or $\frac{32}{20}$. These conclusions follow immediately from our idea of ratio.

Ex. Which is greater $a+x : a-x$ or $a^2+x^2 : a^2-x^2$?

$$a+x : a-x > \text{ or } < a^2+x^2 : a^2-x^2,$$

$$\text{according as } \frac{a+x}{a-x} > \text{ or } < \frac{a^2+x^2}{a^2-x^2},$$

$$\text{or, } \frac{a^2+x^2+2ax}{a^2-x^2} > \text{ or } < \frac{a^2+x^2}{a^2-x^2};$$

and since the former is the greater by the quantity $\frac{2ax}{a^2-x^2}$;

$$\therefore a+x : a-x > a^2+x^2 : a^2-x^2.$$

226. Def. A ratio is called a ratio *of greater inequality*, *of less inequality*, or *of equality*, according as the antecedent is *greater* than, *less* than, or *equal* to, the consequent.

227. *A ratio of greater inequality is diminished, and of less inequality increased, by adding any quantity to both its terms.*

If 1 be added to the terms of the ratio $7 : 4$, it becomes the ratio $8 : 5$, which is less than the former, (Art. 225). And in general, let x be added to the terms of the ratio $a : b$, and it becomes $a + x : b + x$, which is greater or less than the former, according as $\frac{a+x}{b+x} >$ or $< \frac{a}{b}$; or, by reducing them to a common denominator, according as $\frac{ab+bx}{b(b+x)} >$ or $< \frac{ab+ax}{b(b+x)}$; that is, as $bx >$ or $< ax$, or as $b >$ or $< a$.

228. Cor. Hence a ratio of greater inequality is increased, and of less inequality diminished, by taking from the terms a quantity less than either of them.

229. Def. If the antecedents of any ratios be multiplied together, and also the consequents, a new ratio results, which is said to be compounded of the former. Thus $ac : bd$ is said to be compounded of the two $a : b$ and $c : d$. It is also sometimes called the sum of the ratios; and when the ratio $a : b$ is compounded with itself, the resulting ratio, $a^2 : b^2$, is called the double of the ratio $a : b$; and if three of these ratios be compounded together, the result, $a^3 : b^3$, is called the triple of the first, &c. Also the ratio $a : b$ is said to be one third of the ratio $a^3 : b^3$; and $a^{\frac{1}{m}} : b^{\frac{1}{m}}$ is said to be an m^{th} part of the ratio $a : b$.

230. Cor. Let the first ratio be $a : 1$; then $a^2 : 1$, $a^3 : 1$, $\ldots a^n : 1$, are twice, three times, ...n times the first ratio; where n, the index of a, shews what multiple, or part, of the ratio $a^n : 1$, the first ratio $a : 1$ is. On this account the indices $1, 2, 3, \ldots n$, are called *measures* of the ratios $a^1 : 1$, $a^2 : 1$, $a^3 : 1$, $\ldots\ldots a^n : 1$.

231. *If the consequent of the preceding ratio be the antecedent of the succeeding one, and any number of such ratios be taken, the ratio which arises from their composition is that of the first antecedent to the last consequent.*

Let $a : b$, $b : c$, $c : d$, be the ratios, the compound ratio is $a \times b \times c : b \times c \times d$, (Art. 229), or, dividing by $b \times c$ (Art. 224), $a : d$; and similarly for any number of ratios.

232. *A ratio of greater inequality, compounded with another, increases it; and a ratio of less inequality diminishes it.*

Let the ratio $x : y$ be compounded with the ratio $a : b$, then the resulting ratio $ax : by >$ or $<$ the ratio $a : b$, according as $\frac{ax}{by} >$ or $< \frac{a}{b}$ (Art. 225); that is, according as $x >$ or $< y$.

233. *If the difference between the antecedent and consequent of a ratio be small when compared with either of them, the double of the ratio, or the ratio of their squares, is nearly obtained by doubling this difference.*

Let $a + x : a$ be the proposed ratio, where x is small when compared with a; then $a^2 + 2ax + x^2 : a^2$ is the ratio of the squares of the antecedent and consequent; but since x is small when compared with a, x^2, or $x \times x$, is small when compared with $2a \times x$, and much smaller than $a \times a$; therefore $a^2 + 2ax : a^2$, or $a + 2x : a$ (Art. 224), will nearly express the ratio of $a^2 + 2ax + x^2 : a^2$.

Thus the ratio of the square of 1001 to the square of 1000 is nearly 1002 : 1000; the real ratio is 1002·001 : 1000, in which the antecedent differs from its approximate value only by one thousandth part of an unit.

234. Cor. Hence the ratio of the square root of $a + 2x$ to the square root of a is the ratio $a + x : a$, nearly; that is, if the difference of two quantities be small with respect to either of them, the ratio of their square roots is nearly obtained by halving that difference.

235. In the same manner it may be shewn that

$$a \pm 3x : a; \quad a \pm 4x : a; \quad a \pm 5x : a; \quad \&c.$$

are nearly equal respectively to the ratios .

$$(a \pm x)^3 : a^3, \quad (a \pm x)^4 : a^4, \quad (a \pm x)^5 : a^5, \quad \&c.$$

Also $a \pm \frac{1}{3}x : a$, $a \pm \frac{1}{4}x : a$, &c. are nearly equal to the ratios $\sqrt[3]{a \pm x} : \sqrt[3]{a}$; $\sqrt[4]{a \pm x} : \sqrt[4]{a}$; &c. if x be small when compared with a.

$$\text{For} \quad (a \pm x)^3 : a^3 = a^3 \pm 3a^2x + 3ax^2 \pm x^3 : a^3$$

$$= 1 \pm \frac{3x}{a} + \frac{3x^2}{a^2} \pm \frac{x^3}{a^3} : 1, \text{ (Art. 224).}$$

Now if x be small when compared with a, $\frac{x}{a}$ is a small fraction; and $\frac{3x^2}{a^2} : \frac{3x}{a} = \frac{x}{a} : 1$, therefore since $\frac{x}{a}$ is small compared with 1, $\frac{3x^2}{a^2}$ is small compared with $\frac{3x}{a}$; *à fortiori* $\frac{x^3}{a^3}$ is very small compared with 1 or with $\frac{3x}{a}$.

Hence, neglecting $\frac{3x^2}{a^2}$ and $\frac{x^3}{a^3}$, which are very small fractions, $1 \pm \frac{3x}{a} : 1$, or $a \pm 3x : a$, is a near approximation to $(a \pm x)^3 : a^3$, if x be small when compared with a.

Similarly it may be shewn that $a \pm 4x : a$; $a \pm 5x : a$; &c. are approximations respectively to $(a \pm x)^4 : a^4$, $(a \pm x)^5 : a^5$, &c.

Again, since $\sqrt{a \pm x} : \sqrt{a} = \sqrt{1 \pm \frac{x}{a}} : 1$, (Art. 224).

$$= 1 \pm \tfrac{1}{2}\frac{x}{a} - \tfrac{1}{8}\frac{x^2}{a^2} \pm \&c. : 1, \text{ (Art. 152. Ex. 3).}$$

by the same reasoning this ratio is reduced to

$$1 \pm \tfrac{1}{2}\frac{x}{a} : 1, \quad \text{or} \quad a \pm \tfrac{1}{2}x : a.$$

Also $\sqrt[3]{a \pm x} : \sqrt[3]{a} = a \pm \frac{1}{3}x : a$, nearly; and so on.

The utility of the rules here proved will be sufficiently manifest from the following Examples, when it is observed by what a troublesome process the several proposed ratios would be found without the rules.

Ex. 1. $(1{\cdot}5241)^4 : (1{\cdot}524)^4 = 1{\cdot}5240 + 4 \times 0{\cdot}0001 : 1{\cdot}524$ nearly $= 1{\cdot}5244 : 1{\cdot}524$ nearly.

Ex. 2. $\sqrt[3]{729} : \sqrt[3]{728} = 728\tfrac{1}{3} : 728$ nearly.

Ex. 3. $\sqrt[5]{2134} : \sqrt[5]{2131} = 2131\tfrac{3}{5} : 2131$ nearly.

PROPORTION.

236. Def. Four quantities are said to be *proportionals*, when the first is the same multiple, part, or parts, of the second, that the third is of the fourth; that is, when $\frac{a}{b} = \frac{c}{d}$, the four quantities a, b, c, d, are called proportionals. This is usually expressed by

saying a is to b as c is to d, and is thus represented, $a : b :: c : d$; or, sometimes, $a : b = c : d$.

The terms a and d are called the *extremes*, and b and c the *means*.

237. *When four quantities are proportionals, the product of the extremes is equal to the product of the means.*

Let a, b, c, d, be the four quantities; then, since they are proportionals, $\frac{a}{b} = \frac{c}{d}$ (Art. 236); and by multiplying both sides of the equation by bd, $ad = bc$.

238. Cor. 1. If the first be to the second as the second to the third, the product of the extremes is equal to the square of the mean.

239. Cor. 2. Any three terms in a proportion being given, the fourth may be determined from the equation $ad = bc$; for $d = \frac{bc}{a}$, $c = \frac{ad}{b}$, $b = \frac{ad}{c}$, $a = \frac{bc}{d}$. Hence we have the Single Rule of Three in Arithmetic.

240. *If the product of two quantities be equal to the product of two others, the four are proportionals, making the terms of one product the means, and the terms of the other the extremes.*

Let $xy = ab$, then dividing by ay, $\frac{x}{a} = \frac{b}{y}$,

or $x : a :: b : y$. (Art. 236.)

241. *If* a : b :: c : d, *and* c : d :: e : f, *then also* a : b :: e : f. (Euclid, B. v. Prop. xi.)

Because $\frac{a}{b} = \frac{c}{d}$, and $\frac{c}{d} = \frac{e}{f}$; therefore $\frac{a}{b} = \frac{e}{f}$;

or $a : b :: e : f$.

242. *If four quantities be proportionals, they are also proportionals when taken* inversely. (Euclid, B. v. Prop. B.)

If $a : b :: c : d$, then $b : a :: d : c$. For $\frac{a}{b} = \frac{c}{d}$, and dividing unity by each of these equal quantities, or taking their reciprocals, $\frac{b}{a}=\frac{d}{c}$; that is, $b : a :: d : c$.

243. *If four quantities be proportionals, they are proportionals when taken* alternately. (EUCLID, B. v. Prop. XVI.)

If $a : b :: c : d$, then $a : c :: b : d$.

Because the quantities are proportionals, $\frac{a}{b}=\frac{c}{d}$, and multiplying by $\frac{b}{c}$, $\frac{a}{c}=\frac{b}{d}$, or $a : c :: b : d$.

Unless the four quantities are of the *same kind*, the alternation cannot take place; because this operation supposes the first to be some multiple, part, or parts, of the third.

One line may have to another line the same ratio that one weight has to another weight, but a line has no relation in respect of magnitude to a weight. In cases of this kind, if the four quantities be represented by numbers, or other quantities which are similar, the alternation may take place, and the conclusions drawn from it will be just.

244. *When four quantities are proportionals, the first together with the second is to the second, as the third together with the fourth is to the fourth. This operation is called* componendo. (EUCLID, B. v. Prop. XVIII.)

If $a : b :: c : d$, then also

$$a+b : b :: c+d : d.$$

Because $\frac{a}{b}=\frac{c}{d}$, by adding 1 to each side,

$$\frac{a}{b}+1=\frac{c}{d}+1;$$

$$\text{that is, } \frac{a+b}{b}=\frac{c+d}{d};$$

$$\text{or } a+b : b :: c+d : d.$$

245. *Also,* dividendo, *the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth.* (EUCLID, B. v. Prop. XVII.)

Because $\frac{a}{b} = \frac{c}{d}$, by subtracting 1 from each side,

$$\frac{a}{b} - 1 = \frac{c}{d} - 1;$$

$$\text{that is, } \frac{a-b}{b} = \frac{c-d}{d};$$

$$\text{or } a - b : b :: c - d : d.$$

246. *Again*, convertendo, *the first is to its excess above the cond, as the third to its excess above the fourth.* (EUCLID, . v. Prop. E.)

By the last article, $\frac{a-b}{b} = \frac{c-d}{d}$;

$$\text{and } \frac{b}{a} = \frac{d}{c} \text{ (Art. 242)};$$

$$\therefore \frac{a-b}{b} \times \frac{b}{a} = \frac{c-d}{d} \times \frac{d}{c}; \text{ or, } \frac{a-b}{a} = \frac{c-d}{c};$$

$$\text{that is, } a - b : a :: c - d : c;$$

$$\text{and inversely, } a : a - b :: c : c - d.$$

247. *When four quantities are proportionals, the sum of the irst and second is to their difference, as the sum of the third and ourth is to their difference.*

If $a : b :: c : d$; then $a + b : a - b :: c + d : c - d$.

$$\text{By Art. 244, } \frac{a+b}{b} = \frac{c+d}{d};$$

$$\text{and by Art. 245, } \frac{a-b}{b} = \frac{c-d}{d};$$

$$\therefore \frac{a+b}{b} \div \frac{a-b}{b} = \frac{c+d}{d} \div \frac{c-d}{d} \text{ (Art. 82)};$$

$$\text{or } \frac{a+b}{a-b} = \frac{c+d}{c-d};$$

$$\text{that is, } a + b : a - b :: c + d : c - d.$$

248. *When any number of quantities are proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.* (EUCLID, B. V. Prop. XII.)

$$\text{If } a : b :: c : d :: e : f, \text{ \&c.}$$

$$\text{then } a : b :: a + c + e + \text{\&c.} : b + d + f + \text{\&c.}$$

Because $\frac{a}{b} = \frac{c}{d}$, $ad = bc$; in the same manner, $af = be$; also $ab = ba$; hence $ab + ad + af = ba + bc + be$,

$$\text{or } a(b + d + f) = b(a + c + e);$$

$$\therefore \text{ by Art. 240, } a : b :: a + c + e : b + d + f;$$

and similarly when more quantities are taken.

249. *When four quantities are proportionals, if the first and second be multiplied, or divided, by any quantity, as also the third and fourth, the resulting quantities will be proportionals.*

If $a : b :: c : d$, then will $ma : mb :: \frac{c}{n} : \frac{d}{n}$.

$$\text{For } \frac{a}{b} = \frac{c}{d}; \ \therefore \ \frac{ma}{mb} = \frac{\frac{1}{n} \cdot c}{\frac{1}{n} \cdot d} \text{ (Art. 101);}$$

$$\text{or } ma : mb :: \frac{c}{n} : \frac{d}{n}.$$

250. *If the first and third be multiplied, or divided, by any quantity, and also the second and fourth, the resulting quantities will be proportionals.*

$$\text{For } \frac{a}{b} = \frac{c}{d}; \ \therefore \ \frac{ma}{b} = \frac{mc}{d}; \text{ and } \frac{ma}{\frac{1}{n} \cdot b} = \frac{mc}{\frac{1}{n} \cdot d};$$

$$\text{or } ma : \frac{b}{n} :: mc : \frac{d}{n}.$$

251. COR. Hence, in any proportion, if instead of the second and fourth terms quantities proportional to them be substituted, we have still a proportion. For $\frac{b}{n}$ and $\frac{d}{n}$ are in the same proportion with b and d (Art. 236).

252. *In two ranks of proportionals, if the corresponding terms be multiplied together, the products will be proportionals.*

$$\text{If } a : b :: c : d,$$
$$\text{and } e : f :: g : h,$$
$$\text{then also } ae : bf :: cg : dh.$$

Because $\frac{a}{b} = \frac{c}{d}$, and $\frac{e}{f} = \frac{g}{h}$; therefore $\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}$, or $\frac{ae}{bf} = \frac{cg}{dh}$; that is, $ae : bf :: cg : dh$.

This is called *compounding* the proportions.

The proposition is true if applied to *any number* of proportions.

253. *If four quantities be proportionals, the like powers, or roots, of these quantities, will be proportionals.*

If $a : b :: c : d$, then $\frac{a}{b} = \frac{c}{d}$, and $\frac{a^n}{b^n} = \frac{c^n}{d^n}$; or $a^n : b^n :: c^n : d^n$; where n is whole or fractional.

254. If $a : b :: c : d$, to prove that

$$ma \pm nb : pa \pm qb :: mc \pm nd : pc \pm qd.$$

$$\frac{a}{b} = \frac{c}{d};$$

$$\frac{ma}{b} = \frac{mc}{d};$$

$$\frac{ma}{b} \pm n = \frac{mc}{d} \pm n,$$

$$\text{or } \frac{ma \pm nb}{b} = \frac{mc \pm nd}{d}.$$

$$\text{Similarly } \frac{pa \pm qb}{b} = \frac{pc \pm qd}{d};$$

$$\therefore \frac{ma \pm nb}{b} \div \frac{pa \pm qb}{b} = \frac{mc \pm nd}{d} \div \frac{pc \pm qd}{d},$$

$$\frac{ma \pm nb}{pa \pm qb} = \frac{mc \pm nd}{pc \pm qd};$$

or $ma \pm nb : pa \pm qb :: mc \pm nd : pc \pm qd$.

255. *If three quantities* a, b, c *be in continued proportion, that is,* a : b :: b : c, *then* a : c :: $a^2 : b^2$. (EUCLID, B. v. Def. x.)

$$\text{For } \frac{a}{b} = \frac{b}{c};$$

$$\therefore \frac{a}{c} = \frac{a}{b} \times \frac{b}{c} = \frac{a}{b} \times \frac{a}{b}$$

$$= \frac{a^2}{b^2};$$

or, $a : c :: a^2 : b^2$.

256. *If four quantities are in continued proportion, that is,* a : b :: b : c :: c : d, *then* a : d :: $a^3 : b^3$. (EUCLID, B. v. Def. XI.)

$$\text{For } \frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d}, \text{ and } \frac{a}{b} = \frac{b}{c} = \frac{c}{d};$$

$$\therefore \frac{a}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$$

$$= \frac{a^3}{b^3};$$

or, $a : d :: a^3 : b^3$.

257. The Definition of Proportion here used will not serve for a definition in Geometry, because there is no Geometrical method of representing the quotient of $a \div b$, a and b being any Geometrical magnitudes whatever of the same kind. But such magnitudes may always be *multiplied geometrically;* that is, a line may be produced till it becomes n times its original length—an area, or a solid, may be doubled, trebled, &c.—*geometrically*. Hence the strictness of Geometry requires such a definition as that which is the foundation of Euclid's 5th Book, and which may easily be shewn to follow from the Algebraic Definition. For suppose a, b, c, d to represent four quantities in proportion, according to the Algebraical definition; then

$$\frac{a}{b} = \frac{c}{d},$$

$$\text{and } \frac{m}{n} \cdot \frac{a}{b} = \frac{m}{n} \cdot \frac{c}{d},$$

$$\text{or } \frac{ma}{nb} = \frac{mc}{nd};$$

om which it follows, by the nature of fractions, that if $ma > nb$, then $c > nd$; if $ma = nb$, $mc = nd$; if $ma < nb$, $mc < nd$: and ma, mc, are ıy equimultiples *whatever* of the 1st and 3rd quantities, nb, nd any equi- ultiples *whatever* of the 2nd and 4th. Therefore a, b, c, d are propor- onal also according to the Geometrical Definition.

258. *If two numbers,* a *and* b, *be prime to each other, they* re *the least in that proportion.*

If possible, let $\frac{a}{b} = \frac{c}{d}$, where a and b are prime to each other, ıd respectively greater than c and d. If the latter numbers be not ime to each other, divide them by their greatest common measure. hen divide a by b, and c by d, as in Art. 103; thus,

$$\begin{array}{l} b)\ a\ (m \\ \overline{x)\ b\ (n} \\ \qquad y \end{array} \qquad\qquad \begin{array}{l} d)\ c\ (m \\ \overline{r)\ d\ (n} \\ \qquad s \end{array}$$

nd because $\frac{a}{b} = \frac{c}{d}$, the first quotients m, m, are equal; again, nce $\frac{a}{b} = m + \frac{x}{b}$, and $\frac{c}{d} = m + \frac{r}{d}$, we have $\frac{x}{b} = \frac{r}{d}$, or $\frac{b}{x} = \frac{d}{r}$; also, cause b is greater than d, x is greater than r. In the same anner, $\frac{x}{y} = \frac{r}{s}$, and y is greater than s; and so on, if there be re remainders. Thus the remainder in the latter division will come 1 sooner than the remainder in the former. Let $s = 1$; then $= r$; and y, which is greater than 1, will be a common measure of nd b (Art. 105), which is contrary to the supposition.

Cor. Hence, if $\frac{a}{b} = \frac{c}{d}$, and a and b be prime to each other, and d are equimultiples of a and b.

259. *If* a *and* b *be each of them prime to* c, ab *is prime to* c.

If not, let $ab = mr$, and $c = ms$; then since a and b are prime c, they are respectively *prime* to ms, and therefore to m; and cause $ab = mr$, we have $\frac{a}{m} = \frac{r}{b}$; therefore b is a *multiple* of m

(Art. 258, Cor.), which is absurd, since it was before shewn to be *prime* to m.

Cor. 1. If b be equal to a, then a^2 and c have no common measure; or $\frac{a^2}{c}$ is a fraction in its lowest terms.

Cor. 2. In the same manner, $\frac{a^3}{c}$, $\frac{a^4}{c}$, &c. are fractions in their lowest terms.

Cor. 3. If a, b, and c, be *each* of them prime to d, e, and f, abc is prime to def.

For, if a be prime to d and e, it is prime to de, and if it be prime to de and f, it is prime to def. In the same manner, b and c are prime to def; consequently, abc is prime to def.

Cor. 4. If a be prime to b, a^2 is prime to b^2, and a^3 to b^3, &c.

SCHOLIUM.

260. In the definition of Proportion it is supposed that one quantity is some determinate multiple, part, or parts of another; or that the fraction arising from the division of one by the other, (which expresses the multiple, part, or parts, that the former is of the latter,) is a determinate fraction. This will be the case, whenever the two quantities have any common measure whatever.

Let x be a common measure of a and b, and let $a = mx$, $b = nx$; then $\frac{a}{b} = \frac{mx}{nx} = \frac{m}{n}$, where m and n are whole numbers.

But it sometimes happens that the quantities are *incommensurable*, that is, admit of no common measure whatever, as when one represents the circumference of a circle and the other its diameter; in such cases the value of $\frac{a}{b}$ cannot be exactly expressed by any fraction, $\frac{m}{n}$, whose numerator and denominator are whole numbers; yet a fraction of this kind may be found, which will express its value *to any required degree of accuracy.*

Suppose x to be a measure of b, and let $b = nx$; also let a be
reater than mx but less than $(m + 1)x$; then $\frac{a}{b}$ is greater than $\frac{m}{n}$
ut less than $\frac{m+1}{n}$, or the difference between $\frac{m}{n}$ and $\frac{a}{b}$ is less than
; and as x is diminished, since $nx = b$, n is increased, and $\frac{1}{n}$
iminished; therefore by diminishing x, the difference between $\frac{m}{n}$
nd $\frac{a}{b}$ may be made less than any that can be assigned.

If a and b as well as c and d be incommensurable, and if when
lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, $\frac{c}{d}$ lie also between $\frac{m}{n}$ and $\frac{m+1}{n}$, how-
ver the magnitudes m and n are increased, then $\frac{a}{b}$ is equal to $\frac{c}{d}$.
or, if they are not equal, they must have some assignable dif-
rence; and because each of them lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, this
fference is less than $\frac{1}{n}$; but since n may, by the supposition, be
creased without limit, $\frac{1}{n}$ may be diminished without limit, that is,
may become less than any assignable magnitude; therefore $\frac{a}{b}$
d $\frac{c}{d}$ have no assignable difference; that is, $\frac{a}{b}$ is equal to $\frac{c}{d}$; and all
e preceding propositions, respecting proportionals, are true of the
ur magnitudes, a, b, c, d.

VARIATION.

261. In the investigation of the relation which varying and
pendent quantities bear to each other the conclusions are more
adily obtained by expressing only two terms in each proportion,
an by retaining the four.

But though, in considering the variation of such quantities, two terms only are expressed, it will be necessary for the Learner to keep constantly in mind that four are supposed; and that the operations, by which our conclusions are in this case obtained, are in reality the operations of *proportionals*.

262. Def. 1. One quantity is said to *vary directly* as another, when the two quantities depend wholly upon each other, and in such a manner, that if one be changed, the other is changed *in the same proportion**.

Let A and B be mutually dependent upon each other, in such a way, that if A be changed to any other value a, B must be changed to another value b, such that $A : a :: B : b$; then A is said to *vary directly* as B.

N. B. When it is simply stated that one quantity 'varies' as another it is always meant that the one 'varies *directly*' as the other.

Ex. 1. If a man agrees to work for a certain sum per hour, the amount of his wages *varies as* the number of hours during which he works.

Ex. 2. If the altitude of a triangle be invariable, the area varies as the base. For, if the base be increased, or diminished, the area is increased or diminished *in the same proportion*, (the area of a triangle being the half of the rectangle under the base and perpendicular. See Euclid, Book I. Props. 36 and 41.)

The sign $\propto$ placed between two quantities signifies that they vary as each other.

263. Def. 2. One quantity is said to *vary inversely* as another, when the former cannot be changed in any manner, but the reciprocal of the latter is changed *in the same proportion*.

A varies inversely as B, $\left(A \propto \frac{1}{B}\right)$, if, when A is changed to a, B be changed to b, in such a manner that $A : a :: \frac{1}{B} : \frac{1}{b}$; or $A : a :: b : B$.

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* That variation is merely an abridgment of Proportion is a point to be carefully borne in mind; for one quantity is said to "vary" as another, not because the two increase and decrease together, but because as one increases or decreases, the other increases or decreases *in the same proportion*. Thus, if $y = \sqrt{ax}$, in which x and y are varying quantities and a is invariable, y increases as x increases, and diminishes as x diminishes, but y does *not* "*vary*" as x, because, as x increases, y does not increase *in the same proportion*; for instance, if x be doubled, y is not doubled.

Ex. If the area of a triangle be given, the base *varies inversely* as the perpendicular altitude.

Let A and a represent the altitudes, B and b the bases, of two equal triangles; then $\frac{A \times B}{2} = \frac{a \times b}{2}$, or $A \times B = a \times b$; therefore (Art. 240), $A : a :: b : B :: \frac{1}{B} : \frac{1}{b}$.

264. Def. 3. One quantity is said to *vary as two others jointly*, if, when the former is changed in any manner, the product of the other two be changed *in the same proportion*.

Thus A varies as B and C jointly, ($A \propto BC$), when A cannot be changed to a, but the product BC must be changed to bc, such that $A : a :: BC : bc$.

Ex. The area of a triangle varies as its base and perpendicular altitude jointly. Let A, B, P, represent the area, base, and perpendicular altitude of one triangle; a, b, p, those of another; then $A = \frac{1}{2}BP$, and $a = \frac{1}{2}bp$; therefore $\frac{A}{a} = \frac{BP}{bp}$, or $A : a :: BP : bp$.

265. Def. 4. One quantity is said to vary *directly* as a second and *inversely* as a third, when the first cannot be changed in any manner, but the second multiplied by the reciprocal of the third is changed in the same proportion.

A varies directly as B, and inversely as C, $\left(A \propto \frac{B}{C}\right)$, when

$A : a :: \frac{B}{C} : \frac{b}{c}$; A, B, C, and a, b, c, being corresponding values of the three quantities.

Ex. The base of a triangle varies as the area directly and the perpendicular altitude inversely. The notation in the last Article being retained, $\frac{BP}{bp} = \frac{A}{a}$; and multiplying both sides by $\frac{p}{P}$, we have $\frac{B}{b} = \frac{Ap}{aP} = \frac{A}{P} \div \frac{a}{p}$; therefore, $B : b :: \frac{A}{P} : \frac{a}{p}$.

In the following articles, A, B, C, &c. represent corresponding values of any quantities, and a, b, c, &c. any other corresponding values of the same quantities.

266. *If one quantity vary as a second, and that second as a third, the first varies as the third.*

Let $A \propto B$, and $B \propto C$, then shall $A \propto C$. For $A : a :: B : b$, and $B : b :: C : c$, therefore (Art. 241), $A : a :: C : c$; that is, $A \propto C$.

In the same manner, if $A \propto B$, and $B \propto \frac{1}{C}$, then $A \propto \frac{1}{C}$.

267. *If each of two quantities vary as a third, their sum, or difference, or the square root of their product, will vary as the third.*

Let $A \propto C$ and $B \propto C$, then $\overline{A \pm B} \propto C$; also $\sqrt{AB} \propto C$.

$$\begin{aligned} \text{By the supposition, } & A : a :: C : c :: B : b; \\ \therefore\ & A : a :: B : b; \\ \text{alternately } & A : B :: a : b \text{ (Art. 243)}, \\ \text{by composition or division } & A \pm B : B :: a \pm b : b; \\ \text{alternately } & A \pm B : a \pm b :: B : b :: C : c; \\ \text{that is, } & A \pm B \propto C. \end{aligned}$$

$$\begin{aligned} \text{Again, } A &: a :: C : c, \\ \text{and } B &: b :: C : c; \\ \therefore\ AB &: ab :: C^2 : c^2 \text{ (Art. 252)}, \\ \text{and } \sqrt{AB} &: \sqrt{ab} :: C : c \text{ (Art. 253)}; \\ \text{that is, } & \sqrt{AB} \propto C. \end{aligned}$$

268. *If one quantity vary as another, it will also vary as any multiple, or part, of the other.*

Let $A \propto B$, and m be any constant quantity, then because $A : a :: B : b$, $A : a :: mB : mb$, or $A : a :: \frac{B}{m} : \frac{b}{m}$ (Art. 249); that is, $A \propto mB$, or $\propto \frac{B}{m}$.

269. Cor. 1. *If* A *vary as* B, A *is equal to* B *multiplied by some invariable quantity*[*].

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[*] By the application of this rule almost every question in Variation is readily solved, since the variation is convertible into an equation, to which the usual rules may be applied.

For $A : a :: mB : mb$; altern. $A : mB :: a : mb$; if therefore n be so assumed that $A = mB$, then in all cases $a = mb$.

Conversely, if $A = mB$, and m is invariable, then $A \propto B$.

270. Cor. 2. If we know any corresponding values of A and ?, the constant quantity m may be found.

Let a and b be the two values known, then $m = \frac{a}{b}$, and is therefore nown; and, in general, $A = \frac{a}{b} \times B$.

Ex. 1. Let $s \propto t^2$, and when $t = 1$ suppose $s = 16$, then, since $= mt^2$, $16 = m$, and $s = 16t^2$.

Ex. 2. Let $y^2 \propto a^2 - x^2$, and when $x = 0$, suppose $y = b$, then since $= m(a^2 - x^2)$, $b^2 = ma^2$, and $m = \frac{b^2}{a^2}$; $\therefore y^2 = \frac{b^2}{a^2}(a^2 - x^2)$.

271. *If one quantity vary as another, any power or root of e former will vary as the same power or root of the latter.*

Let A vary as B, then $A : a :: B : b$, and by Art. 253, $A^n : a^n$ $: B^n : b^n$; that is, $A^n \propto B^n$, where n is whole or fractional.

272. *If one quantity vary as another, and each of them be ultiplied or divided by any quantity, variable or invariable, the oducts, or quotients, will vary as each other.*

Let A vary as B, and let T be any other quantity. Then, by e supposition, $A : a :: B : b$;

$$\therefore\ AT : at :: BT : bt; \text{ that is, } AT \propto BT.$$

$$\text{Also } \frac{A}{T} : \frac{a}{t} :: \frac{B}{T} : \frac{b}{t}; \text{ that is, } \frac{A}{T} \propto \frac{B}{T}.$$

273. Cor. If $A \propto B$, dividing both by B, $\frac{A}{B} \propto \frac{B}{B} \propto 1$; that $\frac{A}{B}$ is constant.

274. *If one quantity vary as two others jointly, either of the tter varies as the first directly and the other inversely.*

Let $V \propto FT$, then by Art. 272, $F \propto \frac{V}{T}$, or $T \propto \frac{V}{F}$.

275. **Cor.** If the product of two quantities be invariable, those quantities vary inversely as each other.

Let $B \times P$ be constant, or $B \times P \propto 1$; by division, $B \propto \frac{1}{P}$.

276. *If four quantities be always proportionals, and one or two of them be invariable, we may find how the others vary.*

Ex. Let p, q, r, s, be always proportionals, and let p be invariable, then $s \propto qr$. Because $ps = qr$ (Art. 237), $ps \propto qr$; and since p is constant, $s \propto qr$ (Art. 268). If both p and q be invariable, $s \propto r$.

277. *If one quantity vary as a second, and a third as a fourth, the product of the first and third will vary as the product of the second and fourth.*

Let $A \propto B$ and $C \propto D$, then $AC \propto BD$.

$$\begin{aligned} \text{Because } A &: a :: B : b, \\ \text{and } C &: c :: D : d, \\ AC &: ac :: BD : bd \text{ (Art. 252)}; \\ \text{that is, } AC &\propto BD. \end{aligned}$$

278. *When the increase or decrease of one quantity depends upon the increase or decrease of two others, and it appears that if either of these latter be invariable, the first varies as the other; when they both vary, the first varies as their product.*

Let $S \propto V$ when T is given,

and $S \propto T$ when V is given;

when neither T nor V is given, $S \propto TV$.

For the variation of S depends upon the variations of the two quantities T and V; let the changes take place separately, and whilst T is changed to t, let S be changed to S'; then, by the supposition,

$$S : S' :: T : t;$$

but this value S' will again be changed to s, by the variation of V, and in the same proportion that V is changed; that is,

$$S' : s :: V : v;$$

and by compounding this with the last proportion,

$$S'S : S's :: TV : tv;$$

$$\text{or,}\quad S : s :: TV : tv \text{ (Art. 249)},$$

$$\text{that is,}\quad S \propto TV.$$

Ex. If 6 horses can plough 17 acres in 2 days, how many acres will 93 horses plough in $4\frac{1}{2}$ days?

The number of acres ploughed $\propto$ number of horses,

if the days are the same.

.... $\propto$ number of days,

if the horses are the same.

when both horses and days are different,

number of acres $\propto$ number of days $\times$ number of horses;

$\therefore$ number of acres required : 17 acres :: $93 \times 4\frac{1}{2} : 6 \times 2$;

$$\therefore \frac{\text{number of acres required}}{17} = \frac{93 \times 4\frac{1}{2}}{6 \times 2},$$

$$\therefore \text{number of acres required} = \frac{31 \times 2\frac{1}{4} \times 17}{2} = 592\tfrac{7}{8}.$$

279. *If there be any number of magnitudes,* P, Q, R, S, &c., *each of which varies as another,* V, *when the rest are constant; when they are all changed,* V *varies as their product.*

Let $V \propto P$ when Q, R, and S, are given;
$V \propto Q$ when P, R, and S, are given;
$V \propto R$ when P, Q, and S, are given;
$V \propto S$ when P, Q, and R, are given;
when P, Q, R, S, are all variable, $V \propto PQRS$.

Let the changes of V dependent upon the changes of P, Q, R, S take place separately, and whilst P is changed to p, let V be changed to V_1; when Q is changed to q, let V_1 be changed to V_2; when R is changed to r, let V_2 be changed to V_3; and when S is changed to s, let V_3 be changed to v. Then, by the supposition, these changes are such, that we have the following proportions:—

$$V : V_1 :: P : p,$$
$$V_1 : V_2 :: Q : q,$$
$$V_2 : V_3 :: R : r,$$
$$V : v :: S : s;$$
$$\therefore V : v :: PQRS : pqrs; \text{ (Arts. 252, 249)},$$
$$\text{that is,}\quad V \propto PQRS.$$

And the same proof may be extended to any number of magnitudes.

ARITHMETICAL PROGRESSION.

280. **Def.** Quantities are said to be in *Arithmetical Progression* when they increase or decrease by a *Common Difference*. Thus each of the following series of quantities

$$1,\quad 3,\quad 5,\quad 7,\quad 9,\quad \&\text{c.}$$
$$a,\quad a+b,\quad a+2b,\quad a+3b,\quad \&\text{c.}$$
$$a,\quad a-b,\quad a-2b,\quad a-3b,\quad \&\text{c.}$$

is in Arithmetical Progression.

Hence it is manifest, that, if a be the first term and $a + b$ the second, $a + 2b$ is the third, $a + 3b$ the fourth, &c. and $a + (n - 1)b$ the n^{th} term.

By giving the proper value to n any one term may be found independently of the rest. Thus, if the 50th term of the first series be required, we have

$$a = 1,\ b = 2,\ \text{and } n = 50,$$
$$\therefore \text{ the } 50^{\text{th}} \text{ term} = 1 + (50 - 1)\,2,$$
$$= 1 + 98 = 99.$$

281. *If the two extremes, and the number of terms, in an Arithmetical Progression be given, the means, that is, the intervening terms, may be found.*

Let a and l be the extremes, that is, the first and last terms, of an Arithmetical Progression, n the number of terms, and b the unknown common difference; then the progression is $a,\ a+b,\ a+2b,\ \ldots\ldots\ a+\overline{n-1}\,.\,b$; and since l is the last term,

$$a + \overline{n-1}\,.\,b = l;$$
$$\therefore\ b = \frac{l-a}{n-1};$$

and b being thus found, all the means $a + b$, $a + 2b$, &c. are known.

Ex. There are four means, or intervening terms, in Arithmetical Progression between 1 and 36; find them.

Here $a = 1$, $l = 36$, $n = 6$,

$$\therefore\ b = \frac{36-1}{5} = 7;$$

and the means are 8, 15, 22, 29.

Cor. A single Arithmetical mean between any two quantities a and b, which is called *the Arithmetical mean,* will be $a + \frac{b-a}{3-1}$ or $\frac{a+b}{2}$: but this may be shewn more simply thus, Let a, x, b be in Arithmetical Progression,

then by Def. $x - a = b - x$; $\therefore x = \frac{1}{2}(a + b)$.

282. *The sum of a series of quantities in Arithmetical Progression may be found by multiplying the sum of the first and last terms by half the number of terms* [*].

Let a be the first term, b the common difference [†], n the number of terms, l the last term, and s the sum of the series:

$$\text{Then,} \quad a + (a + b) + (a + 2b) + \dots\dots\dots + l = s;$$

also, reversing the order of the terms,

$$l + (l - b) + (l - 2b) + \dots\dots\dots + a = s;$$

$$\text{adding,} \quad (a + l) + (a + l) + (a + l) + \dots \text{to } n \text{ terms} = 2s,$$

$$\text{that is,} \quad n \text{ times } a + l, \text{ or } (a + l)n = 2s;$$

$$\therefore s = (a + l)\frac{n}{2}.$$

COR. 1. Since $l = a + (n - 1)b$, we have also

$$s = \{2a + (n - 1)b\}\frac{n}{2}.$$

COR. 2. Any three of the quantities s, a, n, b, being given, the fourth may be found from the equation

$$s = \{2a + (n - 1)b\} \times \frac{n}{2}.$$

Ex. 1. To find the sum of 14 terms of the series 1, 3, 5, 7, &c.

$$\text{Here } a = 1, \; b = 2, \; n = 14;$$

$$\therefore s = (2 + 26) \times 7 = 196.$$

Ex. 2. Required the sum of 9 terms of the series 11, 9, 7, 5, &c.

$$\text{In this case } a = 11, \; b = -2, \; n = 9;$$

$$\therefore s = (22 - 16) \times \frac{9}{2} = 6 \times \frac{9}{2} = 27.$$

Ex. 3. If the first term of an arithmetical progression be 14, and the sum of 8 terms be 28, what is the common difference, and the series?

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[*] By this rule we are enabled to find the sum of *any number* of terms in Arithmetical Progression, without the trouble of adding them all together.

[†] The *Common Difference* in any proposed case is obviously found by subtracting *any* term from the term next following.

$$\text{Since}\quad \{2a+(n-1)b\}\times\frac{n}{2}=s,$$

$$2a+(n-1)b=\frac{2s}{n},$$

$$(n-1)b=\frac{2s}{n}-2a=\frac{2s-2an}{n};$$

$$\therefore\ b=\frac{2s-2an}{n(n-1)}.$$

In the case proposed $s=28$, $a=14$, $n=8$;

$$\therefore\ b=\frac{56-224}{8\times7}=\frac{7-28}{7}=-3.$$

Hence the series is 14, 11, 8, 5, &c.

Ex. 4. If the first term of an arithmetical progression be $3\frac{1}{3}$, the common difference $1\frac{4}{9}$, and the sum 22, find the number of terms.

$$\text{Generally}\quad s=(2a+\overline{n-1}\,.\,b)\frac{n}{2}.$$

$$\text{Here}\quad s=22,\ a=3\tfrac{1}{3},\ b=1\tfrac{4}{9},$$

$$\therefore\ 22=\left\{\frac{20}{3}+\overline{n-1}\,.\,\frac{13}{9}\right\}.\frac{n}{2}.$$

$$\text{or}\quad 396=47n+13n^2;$$

and by solving this quadratic equation it is found that n, the number of terms, is 4.

In this and similar examples the *negative* value of n is excluded by the supposition under which n originally enters the equation.

283. *The sum of any two terms taken equidistant from the beginning and end of an arithmetical progression is equal to the sum of the first and last terms.*

This is proved in the demonstration of Art. 282; or it may be shewn independently, thus:—Let a, b, c, d, e, f, g be in arithmetical progression, then, by Definition,

$$b-a=g-f,\ \text{or}\ b+f=a+g.$$

Again, $c-b=f-e$; $\therefore\ c+e=b+f=a+g$. And so on, whatever the number of terms in the series may be.

284. *If the number of terms of an arithmetical progression be odd, twice the middle term is equal to the sum of the first and last terms.*

Let q represent the middle term, p the preceding, and r the succeeding term, then $q-p=r-q$, by Definition;

$$\therefore\ 2q=p+r,$$

d p and r are equidistant terms from the beginning and end;

$$\therefore\ p + r = a + l, \text{ by last Art.}$$
$$\text{and } 2q = a + l.$$

285. By the last two articles the summation of series in arithmetical rogression is sometimes capable of being facilitated. Thus, if the number of terms whose sum is required be even, and more than half that number of terms are given, it is sometimes easier to add together two terms uidistant from the beginning and end of the series, than to find either e "common difference" or the last term.

If the number of terms be odd, and the middle term is one of those hich are given, then, from what has been proved, *the middle term multiied by the whole number of terms will be the sum required.*

Ex. 1. Required the sum of $\frac{2}{3} + \frac{7}{15} + \frac{4}{15} + \frac{1}{15} +$ &c. to 7 terms.

Here $\frac{1}{15}$ is the middle term; $\therefore$ sum required $= \frac{1}{15} \times 7 = \frac{7}{15}$.

Ex. 2. Required the sum of $1 + \frac{3}{2} + 2 + \frac{5}{2} +$ &c. to 6 terms.

Here the 3d and 4th terms are equidistant from the beginning and end;

$$\therefore\ \text{sum} = \left(2 + \frac{5}{2}\right) \times 3 = \frac{27}{2} = 13\tfrac{1}{2}.$$

GEOMETRICAL PROGRESSION.

286. Def. Quantities are said to be in *Geometrical Progression*, continual proportion, when the first is to the second as the second the third, and as the third to the fourth, &c. that is, when every ceeding term is a certain multiple, or part, of the preceding term.

If a be the first term, and ar the second, the series will be a, , ar^2, ar^3, ar^4, &c., the n^{th} term being ar^{n-1}.

$$\text{For}\quad a : ar :: ar : ar^2 :: ar^2 : ar^3, \text{ \&c.}$$
$$\text{or,}\quad \frac{ar}{a} = \frac{ar^2}{ar} = \frac{ar^3}{ar^2} = \text{\&c.}$$

287. The constant multiplier, by which any term is derived m the preceding, is called the *Common Ratio*, and it may be und by dividing the second term by the first, or *any other term by at which precedes it.*

Ex. 1. 1, 3, 9, 27, &c. are in Geometrical Progression, find the *ommon Ratio.*

$$\text{Here Common Ratio} = \frac{3}{1} = 3.$$

Ex. 2. $2\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{10}$, &c. are in Geometrical Progression, find the Common Ratio.

$$\text{Here Common Ratio} = \frac{1}{10} \div \frac{1}{2} = \frac{1}{5}.$$

288. *If any terms be taken at equal intervals in a geometrical progression, they will be in geometrical progression.*

Let a, $ar \ldots ar^n \ldots\ldots ar^{2n} \ldots . ar^{3n} \ldots .$&c. be the progression, then a, ar^n, ar^{2n}, ar^{3n}, &c. are at the interval of n terms, and form a geometrical progression, whose common ratio is r^n.

289. *If the two extremes, and the number of terms, in a geometrical progression be given, the means, that is, the intervening terms, may be found.*

Let a and l be the extremes, n the number of terms, and r the common ratio; then the progression is a, ar, ar^2, $ar^3 \ldots ar^{n-1}$; and since l is the last term,

$$ar^{n-1} = l, \text{ and } r^{n-1} = \frac{l}{a};$$

$$\therefore\ r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}};$$

and r being thus known, all the means, or intervening terms, ar, ar^2, ar^3, &c. are known.

Cor. A single Geometrical mean between a and b, called *the Geometrical mean,* will be $a\left(\frac{b}{a}\right)^{\frac{1}{2}}$ or $\sqrt{ab}$; but this may be shewn more simply thus,

Let a, x, b be in Geometrical Progression, theh by Def. $\frac{x}{a} = \frac{b}{x}$;

$$\therefore\ x^2 = ab, \text{ and } x = \sqrt{ab}.$$

Ex. There are three means, or intervening terms, in a Geometrical Progression between 2 and 32; find them.

$$\text{Here } n = 5, \quad a = 2, \quad \text{and} \quad l = 32;$$

$$\therefore\ r = \left(\frac{l}{a}\right)^{\frac{1}{n-1}} = (16)^{\frac{1}{4}} = 2;$$

and the means are 4, 8, 16.

290. *To find the sum of a series of quantities in Geometrical Progression.*

Let a be the first term, r the common ratio, n the number of erms, and s the sum of the series:

$$\text{Then} \quad s = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1},$$

$$\text{and} \quad rs = \quad ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1} + ar^n,$$

$$\text{Subtracting} \quad rs - s = ar^n - a;$$

$$\therefore\ s = \frac{ar^n - a}{r - 1} = a \cdot \frac{r^n - 1}{r - 1}.$$

Cor. 1. If l be the last term, $l = ar^{n-1}$,

$$\therefore\ s = \frac{rl - a}{r - 1}\text{*};$$

from which equation, any three of the quantities s, r, l, a, being given, the fourth may be found.

Cor. 2. When r is a proper fraction, as n increases, the value of r^n, or of ar^n, decreases, and when n is increased without limit, ar^n becomes less, with respect to a, than any magnitude that can be assigned†. Hence the sum of the series, which in general is equal to $\frac{ar^n - a}{r - 1}$ or $\frac{ar^n}{r - 1} - \frac{a}{r - 1}$, is reduced in this case, to $\frac{-a}{r - 1}$, that is, $\frac{a}{1 - r}$.

This quantity $\frac{a}{1 - r}$, which we call the *sum* of the series, is the *limit* to which the sum of the terms approaches, but never actually attains‡; it is however the true representative of the series con-

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* The following is another method of arriving at the same result equally simple:—
Let a, b, c, d, &c., h, k, l, be the series, s the sum, and r the common ratio; then, by definition,

$$b = ar, \quad c = br, \quad d = cr, \ldots k = hr, \quad l = kr,$$

$$\therefore\ b + c + d + \ldots + k + l = (a + b + c + \ldots + h + k)r,$$

$$\text{or} \quad s - a = (s - l)r = sr - rl;$$

$$\therefore\ s = \frac{rl - a}{r - 1}.$$

† Thus, if $r = \frac{3}{10}$ or 0·3, $r^2 = 0{\cdot}09$, $r^3 = 0{\cdot}0027$; and so on, shewing that as n increases r^n decreases, and that such a power of r may be taken as to produce a quantity less than any number which shall be named, however small.

‡ That is, although no *definite* number of terms will amount to $\frac{a}{1-r}$, yet, by taking a sufficient number, the sum will reach as near as we please to it; and, whatever number be taken, their sum will not exceed it.

tinued *sine fine*, for this series arises from the division of a by $\overline{1-r}$; and therefore $\frac{a}{1-r}$ may without error be substituted for it.

Ex. 1. To find the sum of 20 terms of the series, 1, 2, 4, 8, &c.

Here $a = 1$, $r = 2$, $n = 20$;

$$\therefore\ s = \frac{1 \times 2^{20} - 1}{2 - 1} = 2^{20} - 1.$$

Ex. 2. Required the sum of 12 terms of the series, 64, 16, 4, &c.

Here $a = 64$, $r = \frac{1}{4}$, $n = 12$;

$$\therefore\ s = 64 \,.\, \frac{\frac{1}{4^{12}} - 1}{\frac{1}{4} - 1} = \frac{4^3}{4^{11}} \,.\, \frac{4^{12} - 1}{4 - 1} = \frac{1}{4^8} \,.\, \frac{4^{12} - 1}{3}.$$

Ex. 3. Required the sum of 12 terms of the series 1, -3, 9, -27, &c.

Here $a = 1$, $r = -3$, $n = 12$;

$$\therefore\ s = \frac{(-3)^{12} - 1}{-3 - 1} = -\frac{3^{12} - 1}{4}.$$

Ex. 4. To find the sum of the series $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} +$ &c. *in infinitum.*

Here $a = 1$, $r = -\frac{1}{2}$; $\therefore\ s = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$.

291. Recurring decimals are made up of quantities in geometrical progression, where $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. is the common ratio, according as one, two, three, &c. figures recur; and the vulgar fraction, corresponding to such a decimal, is found by summing the series.

Ex. 1. Required the vulgar fraction which is equivalent to the decimal ·123123123 &c.

Here the series is $\frac{123}{10^3} + \frac{123}{10^6} + \frac{123}{10^9} + \ldots\ldots$ *in infinitum*; $a = \frac{123}{10^3}$, and $= \frac{1}{10^3}$;

$$\therefore\ s = \frac{\frac{123}{10^3}}{1 - \frac{1}{10^3}} = \frac{123}{999} = \frac{41}{333}.$$

Or, as follows:—

Let ·123123123 &c. $= s$; then, as in Art. 290, multiply both les by 1000; and 123·123123123 &c. $= 1000s$, and by subtracting the rmer equation from the latter, $123 = 999s$;

$$\therefore\ s = \frac{123}{999} = \frac{41}{333}.$$

Ex. 2. Required the vulgar fraction which is equivalent to $\cdot PPP$ &c. here P contains p digits recurring *in inf.*

$$\begin{aligned} \text{Let}\quad & s = \cdot PPP\ \text{\&c.} \\ \text{then}\quad & 10^p s = P\cdot PPP\ \text{\&c.} \\ \therefore\quad & (10^p - 1)s = P \\ \text{and}\quad & s = \frac{P}{10^p - 1}. \end{aligned}$$

Ex. 3. Required the vulgar fraction equivalent to $\cdot PQQQ$ &c., where contains p digits, and Q contains q digits recurring *in inf.*

$$\begin{aligned} \text{Let}\quad & s = \cdot PQQQ\ \text{\&c.}, \\ \text{then}\quad & 10^{p+q} s = PQ\cdot QQ\ \text{\&c.}, \\ \text{and}\quad & 10^p s = P\cdot QQ\ \text{\&c.}; \\ \therefore\quad & (10^{p+q} - 10^p)s = PQ - P, \\ \text{and}\quad & s = \frac{PQ - P}{10^p(10^q - 1)}. \end{aligned}$$

Both these results may be easily verified by expressing the proposed uantities in geometric progressions—the former being

$$\frac{P}{10^p} + \frac{P}{10^{2p}} + \frac{P}{10^{3p}} + \ldots\ldots \textit{ in inf.}$$

he latter being

$$\frac{P}{10^p} + \frac{Q}{10^{p+q}} + \frac{Q}{10^{p+2q}} + \frac{Q}{10^{p+3q}} + \ldots\ldots \textit{ in inf.}$$

hich may be summed by the rule in Art. 290, Cor. 2.

292. *In a Geometrical series continued* in inf. *any term* $>$, $=$, *or* $<$, *the sum of all that follow, according as the common ratio* $<$, $=$, *or* $> \frac{1}{2}$.

Let $a + ar + ar^2 + \ldots ar^{n-1} + ar^n + \ldots\ldots$ be the series. Then the sum of the series after n terms is the sum of

$$ar^n + ar^{n+1} + \ldots\ldots \text{ which} = ar^n \cdot \frac{1}{1-r}, \text{ (Art. 290.)}$$

$$\text{and } ar^{n-1} >, =, \text{ or } <, ar^n \cdot \frac{1}{1-r},$$

$$\begin{aligned} &\text{according as} && 1 >, =, \text{ or } <, \frac{r}{1-r}, \text{ (Art. 219.)} \\ &\ldots\ \ldots && 1 - r >, =, \text{ or } <, r, \text{ (Art. 219.)} \\ &\ldots\ \ldots && 1 >, =, \text{ or } <, 2r, \text{ (Art. 217.)} \\ &\ldots\ \ldots && r <, =, \text{ or } >, \frac{1}{2}. \text{ (Art. 219).} \end{aligned}$$

HARMONICAL PROGRESSION.

DEF. Any magnitudes A, B, C, D, E, &c. are said to be in *Harmonical Progression*, if $A : C :: A - B : B - C$; $B : D :: B - C : C - D$; $C : E :: C - D : D - E$; &c.

293. *The reciprocals of quantities in Harmonical Progression are in Arithmetical Progression.*

Let A, B, C, &c. be in Harmonical Progression;

then by Def. $A : C :: A - B : B - C$;

$\therefore AB - AC = AC - BC$, (Art. 237),

and dividing both sides by ABC,

$$\frac{1}{C} - \frac{1}{B} = \frac{1}{B} - \frac{1}{A}.$$

Again, $B : D :: B - C : C - D$;

$$\therefore BC - BD = DB - DC,$$

and dividing by BCD,

$$\frac{1}{D} - \frac{1}{C} = \frac{1}{C} - \frac{1}{B};$$

and $\frac{1}{C} - \frac{1}{B}$ has been proved equal to $\frac{1}{B} - \frac{1}{A}$; therefore the quantities $\frac{1}{A}, \frac{1}{B}, \frac{1}{C}, \frac{1}{D}$, have a common difference; that is, they are in Arithmetical Progression. And the same proof may be extended to any number of terms.

Hence every series of quantities in Harmonical Progression may be easily converted into an Arithmetical Progression, and then the rules of Arithmetical Progression may be applied to it. Thus,

Ex. *Given* a *and* b *the first two terms of an Harmonic Series, to find the* n*th term.*

Since $\frac{1}{a}, \frac{1}{b}$ are contiguous terms of an Arithmetic Series,

$$\therefore \text{ the Common Difference of this Series} = \frac{1}{b} - \frac{1}{a},$$

$$\text{and the } n^{\text{th}} \text{ term} = \frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right), \text{ (Art. 280.)}$$

$$= \frac{1}{a} + (n-1)\frac{a-b}{ab} = \frac{(n-1)a - (n-2)b}{ab};$$

$$\therefore \text{ the } n^{\text{th}} \text{ term required} = \frac{ab}{(n-1)a - (n-2)b}.$$

294. *The two extremes, and the number of terms, in an Harmonical Progression being given, the means, or intervening terms, may be found.*

Let a and l be the extremes, and n the number of terms; then since the reciprocals of the terms are in Arithmetic Progression, let b be their common difference, and $\frac{1}{l}$ being the n^{th} term of the Arithmetic Progression, we have

$$\frac{1}{l} = \frac{1}{a} + \overline{n-1}\,.\,b, \text{ (Art. 280)}$$

$$b = \left(\frac{1}{l} - \frac{1}{a}\right) \div \overline{n-1}$$

$$= \frac{a-l}{(n-1)al};$$

hence the Arithmetic Means are

$$\frac{1}{a} + \frac{a-l}{(n-1)al}, \quad \frac{1}{a} + \frac{2(a-l)}{(n-1)al}, \quad \frac{1}{a} + \frac{3(a-l)}{(n-1)al}, \text{ \&c.}$$

$$\text{or} \quad \frac{a+(n-2)l}{(n-1)al}, \quad \frac{2a+(n-3)l}{(n-1)al}, \quad \frac{3a+(n-4)l}{(n-1)al}, \text{ \&c.}$$

$\therefore$ the Harmonic Means are the reciprocals of these quantities, viz.

$$\frac{(n-1)al}{a+(n-2)l}, \quad \frac{(n-1)al}{2a+(n-3)l}, \quad \frac{(n-1)al}{3a+(n-4)l}, \text{ \&c.}$$

COR. A single Harmonic mean between a and b, called *the Harmonic Mean*, will be $\frac{(3-1)\,ab}{a+(3-2)\,b}$, or $\frac{2ab}{a+b}$; but this may be shewn more simply thus,

Let a, x, b be in Harmonic Progression,

$$\text{then by Def. } a : b :: a-x : x-b;$$

$$\therefore\ ax-ab=ab-bx, \quad \text{(Art. 237.)}$$

$$(a+b)\,x=2ab$$

$$\therefore\ x=\frac{2ab}{a+b}.$$

OBS. There is no general expression for the *Sum* of an Harmonic Series, since the sum of any number of quantities is not deducible from the sum of their reciprocals.

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295. Series are sometimes proposed for summation which are not actually composed of terms in Arithmetical or Geometrical Progression, but which may be made to depend upon the rules of one or both by arrangement or artifice. Thus,

Ex. 1. Let the sum of n terms of the following series be required,

$$1+5+13+29+61+\&c.$$

The series is equivalent to

$$\overline{4-3}+\overline{8-3}+\overline{16-3}+\overline{32-3}+\&c.$$

$$=4+8+16+\&c. \text{ to } n \text{ terms} - 3\times n$$

$$=4\cdot\frac{2^n-1}{2-1}-3n=4\,(2^n-1)-3n.$$

Ex. 2. To find the sum of n terms of the series

$$1+2x+3x^2+4x^3+\&c.$$

Let S be the sum required; then

$$S=1+2x+3x^2+\ldots\ldots\ldots\ldots\ldots+nx^{n-1},$$

$$\therefore\ xS=\qquad x+2x^2+\ldots\ldots\ldots+(n-1)\,x^{n-1}+nx^n,$$

$$\text{subtracting } S-xS=1+x+x^2+\ldots\ldots\ldots+x^{n-1}-nx^n,$$

$$\text{or}\quad S\,(1-x)=\frac{1-x^n}{1-x}-nx^n,$$

$$\therefore\ S=\frac{1-x^n}{(1-x)^2}-\frac{nx^n}{1-x}$$

$$=\frac{nx^{n+1}-(n+1)\,x^n+1}{(1-x)^2}.$$

Ex. 3. To find the sum of n terms of the series

$$\frac{n-1}{n}\cdot x+\frac{n-2}{n}\cdot x^2+\frac{n-3}{n}\cdot x^3+\&c.$$

um required $=x+x^2+x^3+\dots\dots+x^n-\dfrac{x}{n}\{1+2x+3x^2+\dots+nx^{n-1}\}$

$$=x\cdot\frac{1-x^n}{1-x}-\frac{x}{n}\cdot\frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2}\ (\text{Ex. 2.})$$

$$=\frac{(n-1)x-nx^2+x^{n+1}}{n(1-x)^2}\,.$$

Ex. 4. To find the sum of n terms of the series

$$a^2+(a+b)^2+(a+2b)^2+\&c.$$

Let $A_1, A_2, A_3, \dots A_n$ represent the several terms in order of the series $a+b$, $a+2b$, &c. and S the sum required; then

$$A_2^3-A_1^3=(A_1+b)^3-A_1^3=3A_1^2b+3A_1b^2+b^3,$$

$$A_3^3-A_2^3=(A_2+b)^3-A_2^3=3A_2^2b+3A_2b^2+b^3,$$

$$A_4^3-A_3^3=(A_3+b)^3-A_3^3=3A_3^2b+3A_3b^2+b^3,$$

$$\dots\dots\dots=\dots\dots\dots\dots\dots=\dots\dots\dots\dots\dots\dots$$

$$A_{n+1}^3-A_n^3=(A_n+b)^3-A_n^3=3A_n^2b+3A_nb^2+b^3;$$

by addition,

$$A_{n+1}^3-A_1^3=3(A_1^2+A_2^2+\dots+A_n^2)\,b+3(A_1+A_2+\dots+A_n)\,b^2+nb^3,$$

$$\text{or } (a+nb)^3-a^3=3bS+3\{2a+(n-1)b\}\frac{nb^2}{2}+nb^3,$$

$$\therefore\ S=\frac{1}{3b}\left\{3na^2b+3n^2ab^2+n^3b^3-3nab^2-\frac{3n^2b^3}{2}+\frac{nb^3}{2}\right\}$$

$$=na^2+n^2ab+\frac{n^3b^2}{3}-nab-\frac{n^2b^2}{2}+\frac{nb^2}{6}$$

$$=na(a+\overline{n-1}\,.\,b)+\frac{nb^2}{6}(2n^2-3n+1).$$

Cor. If $a=b=1$, then

$$1^2+2^2+3^2+\dots+n^2=\frac{n^3}{3}+\frac{n^2}{2}+\frac{n}{6}$$

$$=\frac{2n^3+3n^2+n}{6}$$

$$=\frac{(2n+1)n^2+(2n+1)n}{6}$$

$$=\frac{1}{6}\cdot n(n+1)(2n+1).$$

If $a=1$, $b=2$, then

$$1^2+3^2+5^2+\&\text{c. to } n \text{ terms} = 2n^2 - n + \frac{4n^3}{3} - 2n^2 + \frac{2n}{3}$$

$$= \frac{n}{3}(4n^2-1)$$

$$\text{or } = \frac{(2n-1)\,2n\,(2n+1)}{1\,.\,2\,.\,3};$$

and so on, when any other values are given to a and b.

PERMUTATIONS AND COMBINATIONS.

296. Def. The different orders in which any quantities can be arranged are called their *Permutations.*

Thus the permutations of a, b, c, taken two and two together, are ab, ba, ac, ca, bc, cb; taken three and three together are abc, acb, bac, bca, cab, cba.

297. Def. The *Combinations* of quantities are the different collections that can be formed out of them, without regarding the order in which the quantities are placed.

Thus ab, ac, bc, are the combinations of the quantities, a, b, c, taken two and two; ab and ba, though different permutations, forming the same combination.

298. *The number of permutations that can be formed out of* n *quantities, taken two and two together, is* n (n − 1); *taken three and three together, is* n (n − 1) (n − 2).

In n things, a, b, c, d, &c. a may be placed before each of the rest, and thus form $n-1$ permutations; in the same manner, there are $n-1$ permutations in which b stands first; and so of the rest; therefore there are, upon the whole, $n(n-1)$ permutations of this kind, ab, ba, ac, ca, &c.

Again, of $n-1$ things b, c, d, &c. taken two and two together, there are $(n-1)(n-2)$ permutations, by the former part of the article, and by prefixing a to each of these, there are $(n-1)(n-2)$ permutations, taken three and three, in which a stands first; the same may be said of b, c, d, &c. therefore there are, upon the whole, $n(n-1)(n-2)$ such permutations.

Ex. The number of Permutations of 7 things taken three together $=7\times6\times5=210$.

299. *To find the number of permutations of* n *things taken* r *together.*

By Art. 298, the number taken two together $= n(n-1)$

.... three .... $= n(n-1)(n-2)$

Similarly, ... four .... $= n(n-1)(n-2)(n-3)$.

Now, suppose the law, which is here perceived, to hold generally, that is, let the number of permutations of n things a, b, c, d, &c. taken $r-1$ together be

$$n(n-1)(n-2)\ldots\ldots(n-r+2).$$

Then omitting a, it is equally true that the number of permutations of $n-1$ things b, c, d, &c. taken $r-1$ together is, (putting $n-1$ for n),

$$(n-1)(n-2)\ldots\ldots(n-r+1).$$

Prefix a to each of these last permutations, and there will be a set of permutations of n things taken r together in which a stands first in every permutation, the number of them being

$$(n-1)(n-2)\ldots\ldots(n-r+1).$$

The same number may be made of similar permutations in which b stands first; and so also for each of the n quantities a, b, c, d, &c.

Hence the whole number of permutations which can be made of n things taken r together is

$$n(n-1)(n-2)\ldots\ldots(n-r+1),$$

if it be true that the number of permutations of n things taken $r-1$ together is

$$n(n-1)(n-2)\ldots\ldots(n-r+2).$$

That is, if the *assumed* law be true for any value of r, it is *proved* true for the next higher value. But it has been shewn to hold (Art. 298) when $r=2$, and 3; therefore, it is true when $r=4$; and, if for 4, for 5; if for 5, for 6; and so on generally for any number.

COR. The number of permutations of n things taken all together is

$$\begin{aligned} & n(n-1)(n-2)\ldots\ldots(n-n+1) \\ &= n(n-1)(n-2)\ldots\ldots 1, \\ \text{or } &= 1 \cdot 2 \cdot 3 \ldots\ldots n. \end{aligned}$$

Ex. 1. Required the number of different ways in which 6 persons can be arranged at a dinner table.

Number required = number of permutations of 6 things taken all together

$$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

Ex. 2. Required the number of changes which can be rung upon 12 bells.

$$\text{Number required} = 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$
$$= 479001600.$$

300. *The number of Combinations that can be formed out of* n *things, taken two and two together, is* $n\dfrac{n-1}{2}$; *taken three and three together, the number is* $n.\dfrac{n-1}{2}.\dfrac{n-2}{3}$.

The number of permutations in the first case is $n(n-1)$, (Art. 298) but each combination, ab, admits of two permutations, ab, ba; therefore there are twice as many permutations as combinations, or the number of combinations is $n.\dfrac{n-1}{2}$.

Again, there are $n(n-1)(n-2)$ permutations in n things, taken three and three together; and each combination of three things admits of $3 . 2 . 1$ permutations (Art. 299. Cor.); therefore there are $3 . 2 . 1$ times as many permutations as combinations, and consequently the number of combinations is

$$\frac{n(n-1)(n-2)}{1 . 2 . 3}.$$

301. *To find the number of combinations of* n *things taken* r *together.*

Number of permutations of n things taken r together

$$= n(n-1)(n-2)\ldots\ldots(n-r+1), \text{ (Art. 299).}$$

But every combination of r things will make $1 . 2 . 3 \ldots r$ permutations r together (Art. 299. Cor.); therefore there are $1 . 2 . 3 \ldots r$ times as many permutations as combinations: and consequently the number of combinations is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1 . 2 . 3 \ldots\ldots\ldots\ldots\ldots r}.$$

Ex. Required the number of combinations of 24 different letters taken four and four together.

$$\text{Here} \quad n = 24, \quad r = 4,$$

$$\therefore \text{number required} = \frac{24 \times 23 \times 22 \times 21}{1 \times 2 \times 3 \times 4}$$

$$= 23 \times 22 \times 21 = 10626.$$

302. *To find when the number of combinations of a given number of things is the greatest.*

Since the number of combinations of n things taken r together is equal to the number taken $\overline{r-1}$ together multiplied by $\dfrac{n-r+1}{r}$, the number of combinations will go on increasing as r increases, until $\frac{n-r+1}{r}$ first $=$ or < 1. If r be such, that $\frac{n-r+1}{r} = 1$, that is, if $r = \frac{1}{2}(n+1)$; and if n be odd, so that $\frac{1}{2}(n+1)$ is a whole number, then this value of r will give the greatest number of combinations. If r be the least number which makes $\frac{n-r+1}{r}$ less than 1, that is, if r be the first number greater than $\frac{1}{2}(n+1)$, and n be even, so that $\frac{1}{2}(n+1)$ is a fraction, then the whole number next less than $\frac{1}{2}(n+1)$, that is, $\frac{1}{2}n$, will be the value of r, which gives the greatest number of combinations.

Ex. Of six things how many must be taken together that the number of combinations may be the greatest possible?

Here $n = 6$, an even number; $\therefore$ the number (r) to be taken together $= \frac{1}{2}n = 3$; which will give $\frac{6 \times 5 \times 4}{1 \times 2 \times 3}$, or 20, combinations.

303. *The number of combinations of* n *things taken* r *together is the same as the number of combinations of* n *things taken* $\overline{n-r}$ *together.*

The number of combinations taken $\overline{n-r}$ together is

$$\frac{n(n-1)(n-2)\ldots\{n-(n-r)+1\}}{1 \,.\, 2 \,.\, 3 \ldots\ldots\ldots\ldots (n-r)}, \text{ (putting } \overline{n-r} \text{ for } r \text{ in Art. 301)}$$

$$= \frac{n(n-1)(n-2)\ldots(r+1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots (n-r)}$$

$$= \frac{n(n-1)\ldots(n-r+1)(n-r)\ldots(r+1)}{1 \,.\, 2 \ldots\ldots\ldots\ldots r \,.\, (r+1)\ldots(n-r)}, \text{ if } r < n-r,$$

$$\text{or } = \frac{n(n-1)\ldots(r+1)r(r-1)\ldots(n-r+1)}{1 \,.\, 2 \ldots (n-r)(n-r+1)\ldots(r-1)r}, \text{ if } r > n-r,$$

$$= \frac{n(n-1)\ldots(n-r+1)}{1 \,.\, 2 \ldots\ldots\ldots r} \text{ in both cases,}$$

$=$ number of combinations of n things taken r together.

The truth of this proposition will also appear from a very simple consideration, viz. that of n things if r be taken, $n-r$ things will always be left; and for every different parcel containing r things, there will be a different one left containing $n-r$; therefore the *number* of the former parcels must be equal to that of the latter.

Hence in finding the number of combinations taken r together in certain cases, that is, when $r > \frac{1}{2}n$, it will be a shorter operation to find the number taken $\overline{n-r}$ together.

Ex. Required the number of combinations of 20 things taken 18 together.

Number required = number taken 2 together,

$$=\frac{20\times 19}{1\times 2}=10\times 19=190.$$

304. *To find the number of permutations of* n *things taken all together, when the quantities recur.*

Let a recur p times, b recur q times, c recur r times, &c. And let P represent the number of permutations required. Then if all the a's be changed into different letters, these alone will form $1.2.3....p$ permutations instead of one, and out of *each* of the P permutations we should form $1.2.3....p$ permutations; therefore the whole number would be $P.1.2.3....p$. If again, all the b's be changed to different letters, in the same manner, the b's would of themselves form $1.2.3....q$ permutations, and the whole number of permutations would be increased to

$$P.1.2.3....p.1.2.3....q.$$

And so on, till all the quantities are different. But when all are different, the number of permutations is $1.2.3....n$ (Art. 299. Cor.)

$$\therefore\ P.1.2.3....p.1.2.3....q.1.2.3....r.\ \&c.=1.2.3....n\,;$$

$$\text{and } P=\frac{1.2.3....n}{1.2.3....p.1.2.3....q.1.2.3....r.\ \&c.};$$

or, if $\underline{n|}$ represent $1.2.3....n$, for shortness in writing,

$$P=\frac{\underline{|n}}{\underline{|p}\cdot\underline{|q}\cdot\underline{|r}\cdot\&c.}.$$

Ex. Required the number of permutations that can be formed out of the letters of the word "Mississippi."

Here the whole number of letters is 11.

i recurs 4 times
s ...... 4 ......
p ...... 2 ......

$$\therefore\ P=\frac{1.2.3.4.5.6.7.8.9.10.11}{1.2.3.4.1.2.3.4.1.2}$$

$$=5.7.9.10.11=34650.$$

305. *To find the number of combinations of* n *sets of things, containing respectively* p, q, r, *&c. things, one being taken out of each set for each combination.*

1. First, suppose there are two sets of things, containing p and q things respectively; then the number of combinations made by taking one out of each set is clearly p taken q times, or pq.

2. Next, let there be another set introduced containing r things; then each one of these being combined with pq combinations, there will be pqr combinations of three sets of things, one being taken out of each set.

And so on: the number of combinations required being the continued product of the numbers which express the number of things in each set.

Cor. If there be the same number in each set, or $p=q=r=$ &c., then the number of combinations is p^n.

Ex. There are 7 men, 5 women, and 3 boys; required the number of ways in which they can be taken, so as always to have one and no more out of each set.

The number required $= 7\times5\times3 = 105$.

306. *To find the number of combinations of two sets of things, containing respectively* p *and* q *things,* m *being taken out of one set and* n *out of the other for each combination.*

The number of combinations of the first set taken m together is $\frac{p(p-1)\dots(p-m+1)}{1\ .\ 2\dots\dots\dots\dots m}$; and the number of combinations of the second set taken n together is $\frac{q(q-1)\dots(q-n+1)}{1\ .\ 2\dots\dots\dots\dots n}$.

And, to form the combinations required, each of the latter must be combined with each of the former; therefore the number of them will be the product of these two quantities, or

$$\frac{p(p-1)(p-2)\dots(p-m+1)}{1\ .\ 2\ .\ 3\dots\dots\dots\dots m}\times\frac{q(q-1)\dots(q-n+1)}{1\ .\ 2\dots\dots\dots\dots n}.$$

And similarly if there be more than two sets of things, always taking the continued product of the respective numbers of combinations in each set.

Ex. Out of 10 consonants and 3 vowels how many different collections of letters may be made with 4 consonants and 2 vowels in each?

Here $p=10,\ q=3,\ m=4,\ n=2$;

$$\therefore \text{ the number required} = \frac{10\times9\times8\times7}{1\times2\times3\times4}\times\frac{3\times2}{1\times2} = 10\times9\times7 = 630.$$

THE BINOMIAL THEOREM.

307. The method of raising a binomial to any power by repeated multiplication has before been laid down in Art. 140. The same thing may be done much more expeditiously by the following general rule, which is called the *Binomial Theorem*.

Let $x+a$ be the binomial; its n^{th} power is

$$x^n + nax^{n-1} + n\,.\,\frac{n-1}{2}a^2x^{n-2} + n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}a^3x^{n-3} + \&c.$$

Where the index of x, beginning from n, is diminished by unity, and the index of a, beginning from 0, is increased by unity, in every succeeding term. Also the coefficient of each term is found by multiplying the coefficient of the preceding term by the index of x in that term, and dividing by the index of a increased by unity.

$$\text{Thus } (x+a)^6 = x^6 + 6ax^5 + \frac{6.5}{1.2}a^2x^4 + \frac{6.5.4}{1.2.3}a^3x^3 + \frac{6.5.4.3}{1.2.3.4}a^4x^2$$

$$+ \frac{6.5.4.3.2}{1.2.3.4.5}a^5x + \frac{6.5.4.3.2.1}{1.2.3.4.5.6}a^6$$

$$= x^6 + 6ax^5 + 15a^2x^4 + 20a^3x^3 + 15a^4x^2 + 6a^5x + a^6.$$

308. *To investigate the Binomial Theorem for a positive integral index.*

By actual multiplication it appears that

$$(x+a)(x+b) = x^2 + (a+b)x + ab$$

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc$$

$$(x+a)(x+b)(x+c)(x+d) = x^4 + (a+b+c+d)x^3$$
$$+ (ab+ac+bc+ad+bd+cd)x^2$$
$$+ (abc+acd+bcd+abd)x + abcd;$$

and the same law of formation of the continued product is observed to hold whatever be the number of binomial factors, $x+a$, $x+b$, $x+c$, &c., actually multiplied together, viz. That it is composed of a descending series of powers of x, the index of the highest being the number of factors, and the other indices decreasing by 1 in each succeeding term. Also the coefficient of the first term is 1; of the second the sum of the quantities a, b, c, &c.; of the third the sum of the products of every two; of the fourth the sum of the products of every three; and so on; of the last the product of all the n quantities a, b, c, &c.

Suppose, then, this law to hold for n binomial factors, $x+a$, $x+b$, $x+c$,……, $x+k$; so that

$$(x+a)(x+b)(x+c)\ldots(x+k) = x^n + Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \ldots + K$$

$$\text{where } A = a+b+c+\ldots+k$$
$$B = ab+ac+bc+\ldots.$$
$$C = abc+acd+\ldots\ldots$$
$$\&c. = \&c.$$
$$K = abcd\ldots k;$$

introducing a new factor, $x+l$, we have

$$(x+a)(x+b)(x+c)\ldots(x+k)(x+l) = x^{n+1} + (A+l)x^n + (B+lA)x^{n-1} + \ldots + Kl.$$

$$\text{Hence } A+l = a+b+c+\ldots+k+l,$$
$$B+lA = ab+ac+bc+\ldots+al+bl+\ldots+kl,$$
$$\&c. = \&c.$$
$$Kl = abcd\ldots kl;$$

so that, if the law above described holds when n binomial factors are multiplied together, the same law is *proved* to hold for $n+1$ factors. But it has been shewn to hold up to 4 factors, *therefore* it is true for 5; and, if for 5, then also for 6; and so on, generally, for any number whatever.

Now, let $a=b=c=$ &c.,

then $A=a+a+a+$ &c. to n terms $=na$.

$$\left.\begin{array}{l}B=a^2+a^2+\text{\&c. to as many}\\ \text{terms as is equal to the No.}\\ \text{of combinations of } n \text{ things}\\ \text{taken } \textit{two} \text{ together.}\end{array}\right\}=n\,.\,\frac{n-1}{2}\,a^2,$$

$$\left.\begin{array}{l}C=a^3+a^3+\text{\&c. to as many}\\ \text{terms as is equal to the No.}\\ \text{of combinations of } n \text{ things}\\ \text{taken } \textit{three} \text{ together.}\end{array}\right\}=n\,.\,\frac{(n-1)(n-2)}{2\;.\;3}\,a^3,$$

&c. = &c.

$K=a.a.a\ldots$ to n factors $=a^n$.

Also $(x+a)(x+b)(x+c)\ldots(x+k)$ becomes $(x+a)^n$;

$$\therefore\ (x+a)^n=x^n+nax^{n-1}+\frac{n(n-1)}{1\;.\;2}a^2x^{n-2}+\frac{n(n-1)(n-2)}{1\;.\;2\;.\;3}a^3x^{n-3}+\ldots\ldots+a^n.$$

$$\text{Cor.}\ (1+x)^n=1+nx+\frac{n(n-1)}{1\;.\;2}x^2+\frac{n(n-1)(n-2)}{1\;.\;2\;.\;3}x^3+\ldots\ldots+x^n.$$

309. *To prove the Binomial Theorem when the index is fractional or negative.* (*EULER'S PROOF*).

Let the series $1+mx+\frac{m(m-1)}{1\;.\;2}x^2+$ &c. be represented for all values of m, whether positive, or negative, integral, or fractional, by the *symbol* $f(m)$; then it has been shewn that, when m is a positive integer,

$$f(m)=(1+x)^m.$$

It remains to prove that this equation is also true whem m is either fractional or negative.

By the notation assumed

$$1+nx+\frac{n(n-1)}{1\;.\;2}x^2+\text{\&c.}=f(n);$$

therefore, by multiplication, $f(m)\,.\,f(n)=$ the product of the two series, which will evidently be a series of the form

$$1+ax+bx^2+cx^3+\text{\&c.}$$

ascending regularly by the integral powers of x, the coefficients $a, b, c,$ &c. being different combinations of m and n.

Now, although by changing the values of m and n the *values* of a, b, c, &c. are altered, yet their *forms*, that is, the manner in which m and n enter the series will remain the same. Whatever, therefore, be the *forms* of a, b, c, &c. when m and n are positive integers, the same will they be when m and n are fractional or negative. But in the former case

$$f(m)=(1+x)^m,\ f(n)=(1+x)^n;$$

$$\therefore f(m)\,.\,f(n)=(1+x)^{m+n}=1+(m+n)\,x+\frac{(m+n)(m+n-1)}{1\ .\ 2}x^2+\&c.$$

$\because$ $m+n$ is a positive integer,

$$=f(m+n), \text{ by the notation;}$$

$\therefore$ *universally* $f(m)\,.\,f(n)=f(m+n)$.

Since, then, this last equation is true whatever be the values of m and n, for n write $n+p$, and we have

$$f(m+n+p)=f(m)\,.\,f(n+p)=f(m)\,.\,f(n)\,.\,f(p);$$

and proceeding similarly, we have generally

$$f(m+n+p+\&c.)=f(m)\,.\,f(n)\,.\,f(p)\,.\,\&c. \text{ to any number of terms.}$$

Now, let $m=n=p=\&c.=\frac{h}{k}$, h and k being positive integers; then

$$f\left(\frac{h}{k}+\frac{h}{k}+\&c.\right)=f\left(\frac{h}{k}\right).f\left(\frac{h}{k}\right).f\left(\frac{h}{k}\right).\,\&c.$$

and, if the number of terms be k,

$$f\left(\frac{h}{k}+\frac{h}{k}+\&c. \text{ to } k \text{ terms}\right)=f\left(\frac{h}{k}\right).f\left(\frac{h}{k}\right).\,\&c. \text{ to } k \text{ factors,}$$

$$\text{or } f(h)=\left\{f\left(\frac{h}{k}\right)\right\}^k;$$

$$\text{or } (1+x)^h=\left\{f\left(\frac{h}{k}\right)\right\}^k$$

$$\therefore\ (1+x)^{\frac{h}{k}}=f\left(\frac{h}{k}\right)=1+\frac{h}{k}x+\frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1\ .\ 2}x^2+\ldots\ldots \text{ by the notation,}$$

which proves the Theorem for a fractional positive index.

Again, $\because f(m)\,.\,f(n)=f(m+n)$ for all values of m and n, let $n=-m$, then

$$f(m)\,.\,f(-m)=f(m-m)=f(0)$$

$$=1, \because \text{ the assumed series becomes 1 when } m=0.$$

$$\therefore f(-m)=\frac{1}{f(m)}=\frac{1}{(1+x)^m},$$

$$\text{or } (1+x)^{-m}=f(-m)=1+(-m)\,x+\frac{-m\,(-m-1)}{1\;.\;2}\,x^2+\ldots\ldots$$

hich proves the Theorem for a negative index, integral or fractional.

310. The theorem may be easily proved for a negative index as llows, when it has been previously proved for a positive integral and ictional index.

$$+\,x)^{-n}=\frac{1}{(1+x)^n}=\frac{1}{1+nx+\dfrac{n\,(n-1)}{2}\,x^2+\&c.},$$ *n* being positive, integral

fractional, as before proved; which by actual division of the numerator r the denominator becomes

$$1-nx+\frac{n\,(n+1)}{1\;.\;2}x^2-\frac{n\,(n+1)\,(n+2)}{1\;.\;2\;\;.\;\;3}\,x^3+\ldots$$

$$\text{or } 1+(-n)\,x+\frac{-n\,(-n-1)}{1\;.\;2}\,x^2+\frac{-n\,(-n-1)\,(-n-2)}{1\;.\;2\;\;.\;\;3}\,x^3+\ldots$$

llowing the same law as the expansions proved before for positive dices.

311. If $1\,.\,2\,.\,3\ldots\ldots n$ be represented by $\underline{|n}$, *n* being any positive teger, the Binomial Theorem may be written in the following form:—

$$\text{'}+a)^n=\underline{|n}\,.\left\{\frac{x^n}{\underline{|n}}+\frac{a\,x^{n-1}}{\underline{|1}\,.\,\underline{|n-1}}+\frac{a^2x^{n-2}}{\underline{|2}\,.\,\underline{|n-2}}+\ldots\frac{a^{n-1}x}{\underline{|n-1}\,.\,\underline{|1}}+\frac{a^n}{\underline{|n}}\right\}.$$

312. In applying the Binomial Theorem to any proposed case it is ell to observe, that, if each term of the given binomial be of one dimenon, every term of the expansion will be of *n* dimensions, *n* being the dex of the power to which the binomial is raised.

Also, if each term of the proposed binomial be of two dimensions, 'ery term in the expansion will be of $2n$ dimensions; and so on.

And it will be found convenient in practice to reduce the proposed nomial to such a form that it may have 1 for its first term. Thus we duce $(a+b)^n$ to $a^n\left(1+\dfrac{b}{a}\right)^n$, then expand $\left(1+\dfrac{b}{a}\right)^n$, and multiply every rm of the expansion by a^n.

$$\text{x. 1. } (a^2+x^2)^n=\left\{a^2\left(1+\frac{x^2}{a^2}\right)\right\}^n=a^{2n}\times\left(1+\frac{x^2}{a^2}\right)^n$$

$$=a^{2n}.\left(1+n\,.\,\frac{x^2}{a^2}+n\,.\,\frac{n-1}{2}\,.\,\frac{x^4}{a^4}+\frac{n(n-1)\,(n-2)}{1\;.\;2\;\;.\;\;3}\,.\,\frac{x^6}{a^6}+\&c.\right),$$

$$=a^{2n}+na^{2n-2}x^2+n\,.\,\frac{n-1}{2}\,.\,a^{2n-4}x^4+\frac{n(n-1)(n-2)}{1\;.\;2\;\;.\;\;3}\,.\,a^{2n-6}x^6+\&c.$$

Ex. 2. $(1+x)^{\frac{1}{n}} = 1 + \frac{1}{n}x + \frac{1}{n}.\frac{\frac{1}{n}-1}{2}x^2 + \frac{1}{n}.\frac{\frac{1}{n}-1}{2}.\frac{\frac{1}{n}-2}{3}x^3 + \&c.$

$$= 1 + \frac{1}{n}x - \frac{n-1}{2n^2}x^2 + \frac{(n-1)(2n-1)}{2\,.\,3\,.\,n^3}x^3 - \&c.$$

Ex. 3. $(1+x)^{-\frac{1}{n}} = 1 - \frac{1}{n}x + \frac{n+1}{2n^2}x^2 - \frac{(n+1)(2n+1)}{2\,.\,3\,.\,n^3}x^3 + \&c.$

Ex. 4. $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^4 = (a^{\frac{1}{2}})^4 \left\{ 1 + \frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}} \right\}$

$$= a^2 \left\{ 1 + 4\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}} + \frac{4 \times 3}{1 \times 2}\left(\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)^2 + \frac{4 \times 3 \times 2}{1 \times 2 \times 3}\left(\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)^3 + \frac{4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4}\left(\frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right)^4 \right\}$$

$$= a^2 + 4a^{\frac{3}{2}}b^{\frac{1}{2}} + 6ab + 4a^{\frac{1}{2}}b^{\frac{3}{2}} + b^2.$$

Ex. 5. $\frac{1}{(1+x^2)^3} = (1+x^2)^{-3}$

$$= 1 - 3x^2 + \frac{-3 \times -4}{1 \times 2}x^4 + \frac{-3 \times -4 \times -5}{1 \times 2 \times 3}x^6 + \ldots\ldots$$

$$= 1 - 3x^2 + 6x^4 - 10x^6 + \ldots\ldots\ldots$$

Ex. 6. $(ax+by)^{\frac{1}{2}} = (ax)^{\frac{1}{2}}.\left\{ 1 + \frac{by}{ax} \right\}^{\frac{1}{2}}$

$$= (ax)^{\frac{1}{2}} \left\{ 1 + \frac{1}{2}.\frac{by}{ax} + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{1\,.\,2}\left(\frac{by}{ax}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{1\,.\,2\,.\,3}\left(\frac{by}{ax}\right)^3 + \ldots \right\}$$

$$= (ax)^{\frac{1}{2}} + \frac{1}{2}.\frac{by}{(ax)^{\frac{1}{2}}} - \frac{1}{8}.\frac{(by)^2}{(ax)^{\frac{3}{2}}} + \frac{1}{16}.\frac{(by)^3}{(ax)^{\frac{5}{2}}} - \ldots\ldots$$

313. If either term of the binomial be negative, every odd power of that term will be negative; and consequently the signs of the terms, in which those odd powers are found, will be changed.

Ex. 1. $(1-x)^n = 1 - nx + n\frac{n-1}{2}x^2 - \frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3 + \&c.$

Ex. 2. $(a^2 - x^2)^n = a^{2n} - na^{2n-2}x^2 + n\,.\,\frac{n-1}{2}a^{2n-4}x^4 - \&c.$

314. If the index of the power, to which a binomial is to be raised, be negative, and either term of the binomial be negative, then *every* sign in the series is positive.

Ex. 1. $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{1\,.\,2}x^2 + \frac{n(n+1)(n+2)}{1\,.\,2\,.\,3}x^3 + \ldots$

Ex. 2. $(1-x)^{-3} = 1 + 3x + \frac{3\times 4}{1\times 2}x^2 + \frac{3\times 4\times 5}{1\times 2\times 3}x^3 + \ldots\ldots$

$$= 1 + 3x + 6x^2 + 10x^3 + \ldots\ldots\ldots\ldots\ldots\ldots$$

Ex. 3. $\frac{1}{\sqrt{a^2-x^2}} = (a^2-x^2)^{-\frac{1}{2}} = a^{-1}\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}$

$$= a^{-1}\left\{1 + \frac{1}{2}\cdot\frac{x^2}{a^2} + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{1\times 2}\cdot\frac{x^4}{a^4} + \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)}{1\ .\ 2\ .\ 3}\cdot\frac{x^6}{a^6} + \ldots\ldots\right\}$$

$$= \frac{1}{a} + \frac{1}{2}\cdot\frac{x^2}{a^3} + \frac{3}{8}\cdot\frac{x^4}{a^5} + \frac{5}{16}\cdot\frac{x^6}{a^7} + \ldots\ldots\ldots\ \ldots\ldots$$

315. *To find the general term of the expansion of* $(x+a)^n$.

The 1st term is x^n,

2nd .... na^1x^{n-1},

3rd .... $n\cdot\frac{n-1}{2}a^2x^{n-2}$,

4th .... $n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}a^3x^{n-3}$,

which we observe that the coefficient of any term is formed of the pro-ct of the factors $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}$, &c. in number one less than the num-r which expresses the position of the term; therefore the coefficient of r^{th} term will be

$$\frac{n(n-1)(n-2)\ldots\ldots(n-\overline{r-2})}{1\ .\ 2\ .\ 3\ldots\ldots\ldots\ldots\ldots(r-1)}.$$

Also the index of a is always the same as the denominator of the last tor of the coefficient; and the index of x is the difference between n d the index of a; therefore the whole r^{th} term is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1\ .\ 2\ .\ 3\ldots\ldots\ldots\ldots(r-1)}a^{r-1}x^{n-r+1}.$$

By substituting in this expression any proposed number for r any ngle term of the expansion may be found independently of the rest.

Ex. Required the 5th term of $(a^2-b^2)^{12}$.

Here $r=5$, and $n=12$;

$$\therefore \text{ term required} = \frac{12\ .\ 11\ .\ 10\ .\ 9}{1\ .\ 2\ .\ 3\ .\ 4}\cdot(-b^2)^4\cdot(a^2)^8$$

$$= 495a^{16}b^8.$$

316. *If the index of the binomial be a positive integer, every coefficient in the expansion, formed from the index, is a positive integer.*

For the coefficient of the $(r+1)^{\text{th}}$ term is

$$\frac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{1\;.\;2\;\;.\;\;3\ldots\ldots\ldots\ldots r};$$

and this (by Art. 301) is the same as the number of combinations of n things taken r together. Now this latter number, by the nature of the thing, must be a whole number if n and r be positive whole numbers; therefore also

$$\frac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{1\;.\;2\;\;.\;\;3\ldots\ldots\ldots\ldots r} \text{ is a whole number.}$$

A proof of this independent of the theory of Combinations, will be given hereafter.

317. *To find the number of terms in the expansion of a binomial.*

The $(r+1)^{\text{th}}$ term of $(x+a)^n$ is

$$\frac{n\,(n-1)\,(n-2)\ldots(n-r+1)}{1\;.\;2\;\;.\;\;3\ldots\ldots\ldots\ldots r}\,a^r x^{n-r};$$

and if r be such that $n-r+1$ is equal to 0, that is, if n be a positive whole number and $r=n+1$, there is no term after the r^{th}, or the number of terms is $n+1$, that is, greater by 1 than the index.

If n be negative or fractional, since r must necessarily be a positive integer, no value of r can make $n-r+1$ equal to 0, and therefore in these cases the number of terms is unlimited.

Thus the number of terms in the expansions of $(x+a)^3$, $(x+a)^7$, is 4 and 8 respectively; but the number for $(x+a)^{-2}$, or $(x+a)^{\frac{1}{2}}$ is unlimited, or indefinitely great.

318. *To prove that, in an expanded binomial, when the index is a positive integer, the coefficients, formed from the index, of any two terms taken equidistant from the beginning and end, are the same.*

Since the number of terms is $n+1$, the $(r+1)^{\text{th}}$ term from the end is the $(\overline{n+1-r})^{\text{th}}$ or $(n-r+1)^{\text{th}}$ term from the beginning, and its coefficient (Art. 315, putting $n-r+1$ for r,) is

$$\frac{n\,(n-1)\,(n-2)\ldots(n-\overline{n-r-1})}{1\;.\;2\;\;.\;\;3\ldots\ldots\ldots(n-r)}$$

$$=\frac{n\,(n-1)\,(n-2)\ldots(r+1)}{1\;.\;2\;\;.\;\;3\ldots\ldots(n-r)}$$

$$=\frac{n\,(n-1)\,(n-2)\ldots(n-r+1)\,(n-r)\ldots(r+1)}{1\;.\;2\;\;.\;\;3\ldots\ldots\ldots r\;\;.\;\;(r+1)\ldots(n-r)},\ \text{if } r<n-r,$$

$$\text{or }=\frac{n\,(n-1)\,(n-2)\ldots(r+1)\,r\,(r-1)\ldots(n-r+1)}{1\;.\;2\;\;.\;\;3\ldots(n-r)\,(n-r+1)\ldots(r-1)\,r},\ \text{if } r>n-r,$$

$$=\frac{n(n-1)(n-2)\ldots(n-r+1)}{1\;.\;2\;\;.\;\;3\ldots\ldots\ldots r}\text{, in both cases,}$$

= coefficient of the $(r+1)^{\text{th}}$ term from the beginning.

This result is shewn at once by Art. 311. Or it may be obtained by riting a for x, and x for a, in the expansion of $(x+a)^n$, so as to deduce at of $(a+x)^n$, and then equating the coefficients of similar terms in the o expansions, which are clearly equal to each other.

Cor. Hence in expanding a binomial, with the index a positive teger, the latter half of the expansion may be taken from the first half.

Ex. Required to expand $(a+b)^7$.

Here the number of terms is 8; and it will be necessary to calculate e coefficient up to the 4th term only;

$$\therefore\;(a+b)^7=a^7+7a^6b+\frac{7.6}{1.2}a^5b^2+\frac{7.6.5}{1.2.3}a^4b^3+\&\text{c.}$$

$$=a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7.$$

319. *To find the greatest term in the expansion of* $(\mathrm{a}+\mathrm{b})^{\mathrm{n}}$.

The $(r+1)^{\text{th}}$ term of the expansion is $\dfrac{n(n-1)\ldots(n-r+1)}{1\;.\;2\ldots\ldots\ldots r}\,a^{n-r}b^r$;

d the r^{th} term is $\dfrac{n(n-1)\ldots(n-r+2)}{1\;.\;2\ldots\ldots(r-1)}\,.\,a^{n-r+1}b^{r-1}$.

Therefore the $(r+1)^{\text{th}}$ term is obtained from the r^{th} by multiplying e latter by $\dfrac{n-r+1}{r}\,.\,\dfrac{b}{a}$.

Hence the r^{th} term will be the greatest when $\dfrac{n-r+1}{r}\,.\,\dfrac{b}{a}$ *first* <1,

$$\text{or}\quad (n-r+1)b<ar,\ \text{(Art. 219)},$$

$$\text{or}\quad r(a+b)>(n+1)b,\ \text{(Art. 217)},$$

$$\text{or}\quad r>(n+1)\frac{b}{a+b}.\quad\text{(Art. 219.)}$$

Take r, therefore, the *first* whole number greater than $(n+1)\dfrac{b}{a+b}$, d the r^{th} term will be the greatest.

If $(n+1)\dfrac{b}{a+b}$ be a whole number, then two terms are equal, each of ich is greater than any of the other terms.

Cor. By thus ascertaining the greatest term we determine the point m which the terms of a series become less and less, or, as it is usually ted, the point at which the series begins to converge.

Ex. Required to find which is the greatest term in the expansion of $(3+5x)^8$, when $x=\frac{1}{2}$.

$$\text{Here}\quad (n+1)\,\frac{b}{a+b}=(8+1)\,.\,\frac{\frac{5}{2}}{3+\frac{5}{2}}$$

$$=9\times\frac{5}{11}=\frac{45}{11}=4\tfrac{1}{11}.$$

The *first* whole number greater than $4\frac{1}{11}$ is 5; therefore the term required is the 5th.

320. *To find the sum of all the coefficients of an expanded binomial.*

Since $(1+x)^n=1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\&\text{c.}$, for any value whatever of x, let $x=1$, then

$$(1+1)^n=1+n+\frac{n(n-1)}{1\,.\,2}+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}+\&\text{c.}$$

or $2^n=$ the sum of the coefficients.

Ex. $(x+a)^5=x^5+5ax^4+10a^2x^3+10a^3x^2+5a^4x+a^5$, and the sum of the coefficients $=1+5+10+10+5+1=32=2^5$.

321. *To find the approximate roots of numbers by the Binomial Theorem.*

The theorem being proved for a fractional index, we have

$$\sqrt[n]{1\pm x}=(1\pm x)^{\frac{1}{n}}=1\pm\frac{1}{n}x-\frac{1}{n}\,.\,\frac{n-1}{2n}x^2\pm\ldots\ldots$$

Now, if N represent a proposed number whose n^{th} root is required, take p such that p^n is the nearest perfect n^{th} power to N, so that $N=p^n\pm q$, q being small compared with p^n; + or − according as $N>$ or $<p^n$;

$$\text{then}\quad \sqrt[n]{N}=p\left(1\pm\frac{q}{p^n}\right)^{\frac{1}{n}}\ ;\ \text{and writing}\ \frac{q}{p^n}\ \text{for}\ x,$$

$$=p\left\{1\pm\frac{1}{n}\,.\,\frac{q}{p^n}-\frac{1}{n}\,.\,\frac{n-1}{2n}\left(\frac{q}{p^n}\right)^2\pm\ldots\right\},$$

of which series a few terms only will give the required root to a considerable degree of accuracy.

Ex. Required the approximate cube root of 128.

$$\text{Here}\quad \sqrt[3]{128}=\sqrt[3]{5^3+3}=5\sqrt[3]{1+\frac{3}{125}}$$

$$= 5\{1 + \frac{1}{3} \cdot \frac{3}{125} - \frac{1}{3} \cdot \frac{1}{3}\left(\frac{3}{125}\right)^2 + \frac{1}{3^2} \cdot \frac{5}{9}\left(\frac{3}{125}\right)^3 - \dots\}$$

$$= 5 + \frac{1}{5^2} - \frac{1}{5^5} + \frac{1}{3} \cdot \frac{1}{5^7} - \dots$$

$$= 5 + 0{\cdot}04 - 0{\cdot}00032 + 0{\cdot}0000042 - \dots \text{(Art. 57)}$$

$$= 5{\cdot}0396842.$$

322. A trinomial, $a + b + c$, may be raised to any power by considering two terms as one, and making use of the Binomial Theorem. Thus,

$$(a + b + c)^n = (\overline{a + b} + c)^n$$

$$= (a + b)^n + n(a + b)^{n-1}c + \frac{n(n-1)}{1 \, . \, 2}(a + b)^{n-2}c^2 + \&\text{c.}$$

in which the several powers of $\overline{a + b}$ may be replaced by their expansions found by the Binomial Theorem.

Ex. Required the cube of $1 + x + x^2$.

$$(1 + x + x^2)^3 = (\overline{1 + x} + x^2)^3$$

$$= (1 + x)^3 + 3(1 + x)^2x^2 + 3(1 + x)x^4 + x^6$$

$$= 1 + 3x + 3x^2 + x^3 + 3x^2 + 6x^3 + 3x^4 + 3x^4 + 3x^5 + x^6$$

$$= 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6.$$

Similarly $(a + b + c + d)^n$ may be expanded by considering $a + b$ as one term and $c + d$ as another; and any multinomial may be expanded in a similar manner by dividing the whole into two terms and considering it as a binomial.

Also any *particular term* of an expanded trinomial may be easily found:—thus

To find the term involving x^4 in the expansion of $(1 + x + x^2)^3$.

$$(\overline{1 + x} + x^2)^3 = (1 + x)^3 + 3(1 + x)^2x^2 + 3(1 + x)x^4 + x^6,$$

and without farther expansion it is seen that

$$\text{the term required} = 3x^2 \times x^2 + 3x^4 = 6x^4.$$

Again, the *number of terms* in the expansion of $(a + b + c)^n$ may be found; for it will obviously be the aggregate number of the terms in the expansions of the several powers of $\overline{a + b}$, from $(a + b)^n$ to $(a + b)^0$, that is (Art. 317)

$$= (n + 1) + n + (n - 1) + \&\text{c. to } \overline{n + 1} \text{ terms}$$

$$= \{2n + 2 - n\}\frac{n + 1}{2} \text{ (Art. 282)} = \frac{(n + 1)(n + 2)}{1 \, . \, 2}.$$

323. *Required to find the Remainder after taking* r *terms of the expansion of* $(1-x)^{-2}$.

By the Binomial Theorem,

$$(1-x)^{-2}=1+2x+3x^2+\dots+rx^{r-1}+R,$$

R representing the remainder after r terms;

$$\therefore \frac{1}{1-2x+x^2}=1+2x+3x^2+\dots+rx^{r-1}+R;$$

$$\therefore 1=\left\{\begin{array}{l} 1+2x+3x^2+\dots\dots\dots \quad +rx^{r-1}+R \\ \quad -2x-4x^2+\dots-2(r-1)x^{r-1}-2rx^r-2Rx \\ \qquad +\ x^2+\dots+\ (r-2)x^{r-1}+(r-1)x^r+rx^{r+1}+Rx^2 \end{array}\right.$$

$$\text{or}\quad 1=1-(r+1)x^r+rx^{r+1}+R(1-x)^2;$$

$$\therefore R=\frac{(r+1)x^r-rx^{r+1}}{(1-x)^2}.$$

In the same manner may be found the remainder after taking r terms of the expansions of $(1-x)^{-3}$, $(1-x)^{-4}$, &c.

324. *To find the number of homogeneous products of* r *dimensions which can be made of* n *things* a, b, c, d, *&c. and their powers.*

By common division, or the Binomial Theorem,

$$\frac{1}{1-ax}=1+ax+a^2x^2+a^3x^3+\dots\dots$$

$$\frac{1}{1-bx}=1+bx+b^2x^2+b^3x^3+\dots\dots$$

$$\frac{1}{1-cx}=1+cx+c^2x^2+c^3x^3+\dots\dots$$

$$\text{\&c.}\quad=\qquad\qquad \text{\&c.}$$

$$\begin{aligned}\therefore \frac{1}{1-ax}\cdot\frac{1}{1-bx}\cdot\frac{1}{1-cx}\cdot\text{\&c.}&=1+(a+b+c+\text{\&c.})\,x\\ &+(a^2+ab+b^2+ac+bc+c^2+\text{\&c.})x^2\\ &+(a^3+a^2b+ab^2+b^3+a^2c+b^2c+ac^2+c^3+\text{\&c.})x^3\\ &+\ \text{\&c.}\qquad\qquad \text{\&c.}\end{aligned}$$

the coefficient of x^r being the sum of the homogeneous products of n things a, b, c, d, &c. of r dimensions.

Now to obtain the *number* of these products, let $a=b=c=d=\text{\&c.}=1$, then the coefficient of x^r will give the number required. But on this supposition, the left-hand side of the equation becomes $(1-x)^{-n}$, which by the Binomial Theorem is

$$1 + nx + n.\frac{n+1}{2}x^2 + n.\frac{n+1}{2}.\frac{n+2}{3}x^3 + \&c.\ldots$$

$$+\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{1\,.\,2\,.\,3\;\ldots\ldots\;r}x^r + \ldots\ldots;$$

$$\therefore \text{ number required} = \frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{1\,.\,2\,.\,3\;\ldots\ldots\;r}.$$

Cor. Hence also the number of terms in the expansion of any multinomial, as $(a_1 + a_2 + a_3 + \ldots a_r)^n$, is known; for it is obviously the same as the number of homogeneous products of r things taken n together, that is,

$$\frac{r(r+1)(r+2)\ldots\ldots(r+n-1)}{1\,.\,2\,.\,3\;\ldots\ldots\;n}.$$

If $r=2$, that is for a binomial, $(a+b)^n$, the expression becomes

$$\frac{2.3.4\ldots\ldots(n+1)}{1.2.3\ldots\ldots n}, \text{ or } n+1.$$

If $r=3$, the number is for $(a+b+c)^n$

$$\frac{3.4.5\ldots\ldots(n+2)}{1.2.3\ldots\ldots n}, \text{ or } \frac{(n+1)(n+2)}{1\,.\,2}.$$

If $r=4$ or the quantity to be expanded be $(a+b+c+d)^n$, the number of terms is

$$\frac{4.5.6\ldots\ldots(n+3)}{1.2.3\ldots\ldots n}, \text{ or } \frac{(n+1)(n+2)(n+3)}{1\,.\,2\,.\,3},$$

and so on for any value of r.

THE EXPONENTIAL THEOREM.

325. *To expand* a^x *in a series of powers of* x.

$a^x = \{(1+\overline{a-1})^n\}^{\frac{x}{n}}$; and expanding by the Binomial Theorem,

$$= \{1 + n(a-1) + n.\frac{n-1}{2}(a-1)^2 + n.\frac{n-1}{2}.\frac{n-2}{3}(a-1)^3 + \&c.\}^{\frac{x}{n}}$$

$$= \{1 + [(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^5}{3} - \&c.]n + Bn^2 + Cn^3 + \ldots\}^{\frac{x}{n}}$$

B, C, &c. containing powers of $\overline{a-1}$ only;

$$= \{1 + An + Bn^2 + Cn^3 + \&c.\}^{\frac{x}{n}}, \text{ if } a-1-\frac{1}{2}(a-1)^2 + \&c. = A$$

$$= 1 + \frac{x}{n}(An + Bn^2 + \ldots) + \frac{x}{n}.\frac{\frac{x}{n}-1}{2}(An + Bn^2 + \ldots)^2 + \ldots\ldots$$

$$= 1 + x(A + Bn + \ldots) + x.\frac{x-n}{2}(A + Bn + \ldots)^2 + \ldots\ldots$$

Now, since a^x is clearly independent of n, n may be any value whatever; let, then, $n=0$;

$$\therefore\ a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \ldots\ldots$$

Cor. If ϵ be that value of a which makes A equal to 1, then

$$\epsilon^x = 1 + \frac{x}{1} + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots\ldots$$

Hence, making $x=1$,

$$\epsilon = 1 + \frac{1}{1} + \frac{1}{1.2} + \frac{1}{1.2.3} + \ldots = 2{\cdot}7182818 \text{ (Art. 57).}$$

THE MULTINOMIAL THEOREM.

326. The Multinomial Theorem is a rule or formula for expanding any power of a quantity which consists of more than two terms.

The expansion of a multinomial may frequently be effected by the Binomial Theorem, as is done for a trinomial in Art. 322; for $(a+b+c+d+\&c.)^m$ may be expanded as a binomial by considering any number of terms as one term, and the remainder as another term. But a more general method is to find *the general term*, and to deduce the whole expansion from that term as follows:—

327. *To find the general term of the expansion of* $(a+b+c+d+\&c.)^m$.

Let $b+c+d+\&c.=z$, then $(a+b+c+d+\&c.)^m = (a+z)^m$, of which the general term, expressed by the $\overline{\alpha+1}^{\text{th}}$, is

$$\frac{m(m-1)\ldots\ldots(m-\alpha+1)}{1\ .\ 2\ \ldots\ldots\ \alpha}\,a^p z^\alpha, \text{ where } p+\alpha=m,$$

or $m(m-1)\ldots\ldots(p+1)\ .\ a^p\ .\ \dfrac{z^\alpha}{\underline{|\alpha}}$, *where* α *is a positive integer.*

Again, if $c+d+\&c.=y$, then $z=(y+b)^\alpha$, of which the general term, expressed by the $\overline{q+1}^{\text{th}}$, is

$$\frac{\alpha(\alpha-1)\ldots\ldots(\alpha-q+1)}{1\ .\ 2\ \ldots\ldots\ q}\,b^q y^\beta, \text{ (where } q+\beta=\alpha, \text{ or } p+q+\beta=m),$$

or $\dfrac{\alpha(\alpha-1)\ldots(\beta+1)}{1\ .\ 2\ \ldots\ q}\ .\ \dfrac{\beta(\beta-1)\ldots 3\ .\ 2\ .\ 1}{1\ .\ 2\ \ldots\ \beta}\,b^q y^\beta$, ($\beta$ being a pos. integer),

or $\underline{|\alpha}\ .\ \dfrac{b^q}{\underline{|q}}\ .\ \dfrac{y^\beta}{\underline{|\beta}}$; so that the general term of the multinomial becomes

$$m(m-1)\ldots\ldots(p+1)\ .\ a^p\ .\ \frac{b^q}{\underline{|q}}\ .\ \frac{y^\beta}{\underline{|\beta}}\ .$$

Again, if $\overline{d+\&c.}=x$, $y^{\beta}=(x+c)^{\beta}$, of which the general term, expressed by the $\overline{r+1}^{\text{th}}$, is

$$\frac{\beta(\beta-1)\ldots\ldots(\beta-r+1)}{1\,.\,2\,\ldots\ldots\,r}c^{r}x^{\gamma},\ (\text{where } r+\gamma=\beta,\ \text{or } p+q+r+\gamma=m),$$

or $$\frac{\beta(\beta-1)\ldots(\gamma+1)}{1\,.\,2\,\ldots\,r}\cdot\frac{\gamma(\gamma-1)\ldots3\,.\,2\,.\,1}{1\,.\,2\,\ldots\,\gamma}c^{r}x^{\gamma},\ (\gamma \text{ being a pos. integer}),$$

or $\underline{|\beta}\cdot\frac{c^{r}}{\underline{|r}}\cdot\frac{x^{\gamma}}{\underline{|\gamma}}$; so that the general term of the multinomial becomes

$$m(m-1)\ldots\ldots(p+1)\,.\,a^{p}\,.\,\frac{b^{q}}{\underline{|q}}\cdot\frac{c^{r}}{\underline{|r}}\cdot\frac{x^{\gamma}}{\underline{|\gamma}};$$

and so on, until the terms of the multinomial are exhausted.

Hence, the general term required is

$$m(m-1)\ldots\ldots(p+1)\,.\,a^{p}\,.\,\frac{b^{q}}{\underline{|q}}\cdot\frac{c^{r}}{\underline{|r}}\cdot\frac{d^{s}}{\underline{|s}}\cdot\&\text{c.,where } p+q+r+s+\&\text{c.}=m,$$

p being fractional or negative when m is fractional or negative, but q, r, s, &c. always positive integers.

Cor. If m is a positive integer, then, since p is a positive integer, the expression for the general term may be written

$$m(m-1)\ldots(p+1)\,.\,p(p-1)\ldots3\,.\,2\,.\,1\,.\,\frac{a^{p}}{\underline{|p}}\cdot\frac{b^{q}}{\underline{|q}}\cdot\frac{c^{r}}{\underline{|r}}\,.\,\&\text{c.}$$

$$\text{or } \underline{|m}\,.\,\frac{a^{p}}{\underline{|p}}\cdot\frac{b^{q}}{\underline{|q}}\cdot\frac{c^{r}}{\underline{|r}}\,.\,\&\text{c.}$$

The last result may also be arrived at by the following method, assuming *the index a positive integer:*—

328. *To expand* $(a+b+c+d+\&c.)^{m}$, *when* m *is a positive integer.*

$$\epsilon^{(a+b+c+d+\&c.)x}=\epsilon^{ax}\,.\,\epsilon^{bx}\,.\,\epsilon^{cx}\,.\,\epsilon^{dx}\,.\,\&\text{c.}$$

and if $\epsilon=2{\cdot}7182818$, expanding by the Exponential Theorem,

$$1+(a+b+c+d+\&\text{c.})\frac{x}{1}+(a+b+c+d+\&\text{c.})^{2}\frac{x^{2}}{\underline{|2}}+\ldots$$

$$+(a+b+c+d+\&\text{c.})^{m}\frac{x^{m}}{\underline{|m}}+\ldots$$

$$=\left(1+ax+\frac{a^{2}x^{2}}{\underline{|2}}+\frac{a^{3}x^{3}}{\underline{|3}}+\ldots+\frac{a^{m}x^{m}}{\underline{|m}}+\ldots\right)$$

$$\times\left(1+bx+\frac{b^{2}x^{2}}{\underline{|2}}+\frac{b^{3}x^{3}}{\underline{|3}}+\ldots+\frac{b^{m}x^{m}}{\underline{|m}}+\ldots\right)$$

$$\times\left(1+cx+\frac{c^{2}x^{2}}{\underline{|2}}+\frac{c^{3}x^{3}}{\underline{|3}}+\ldots+\frac{c^{m}x^{m}}{\underline{|m}}+\ldots\right)\cdot$$

$$\times\,\&\text{c.}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Now, as this operation merely exhibits the same quantity expanded in two different ways by the same theorem, the corresponding terms, that is, the terms involving the same powers of x will be equal to each other; therefore equating the coefficient of x^m on the one side with the coefficient of x^m on the other, and observing that each separate term on this side of the equation which involves x^m will be the product of as many terms as there are series to be multiplied, one of which is taken out of each series, and will therefore be of the form

$$\frac{a^p x^p}{\lfloor\underline{p}} \cdot \frac{b^q x^q}{\lfloor\underline{q}} \cdot \frac{c^r x^r}{\lfloor\underline{r}} \,.\, \&\text{c. or } \frac{a^p b^q c^r \,.\, \&\text{c.}}{\lfloor\underline{p} \cdot \lfloor\underline{q} \cdot \lfloor\underline{r} \,.\, \&\text{c.}} x^{p+q+r+\&\text{c.}},$$

where $p + q + r + \&\text{c.} = m$, we have

$$\frac{(a+b+c+d+\&\text{c.})^m}{\lfloor\underline{m}} = \Sigma \frac{a^p b^q c^r \,.\, \&\text{c.}}{\lfloor\underline{p} \cdot \lfloor\underline{q} \cdot \lfloor\underline{r} \,.\, \&\text{c.}} \text{[*]},$$

$$\therefore\ (a+b+c+d+\&\text{c.})^m = \Sigma \frac{\lfloor\underline{m}}{\lfloor\underline{p} \cdot \lfloor\underline{q} \cdot \lfloor\underline{r} \,.\, \&\text{c.}} a^p b^q c^r.\ \&\text{c.} \text{[†]}$$

Cor. 1. If $q + r + s + \&\text{c.} = \pi$, then $p = m - \pi$, and the general term becomes

$$\frac{m \,.\, (m-1) \ldots (m-\pi+1)}{1 \,.\, 2 \ldots q \,.\, 1 \,.\, 2 \ldots r \,.\, \&\text{c.}} a^{m-\pi} b^q c^r \,.\, \&\text{c.}$$

which form is sometimes found more convenient.

Cor. 2. If it be required to expand $(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \&\text{c.})^m$, the general term may be obtained from that of $(a + b + c + d + \&\text{c.})^m$ by writing a_0, $a_1 x$, $a_2 x^2$, $a_3 x^3$, &c. in place of a, b, c, d, &c. respectively, by which it becomes

$$\frac{\lfloor\underline{m}}{\lfloor\underline{p} \cdot \lfloor\underline{q} \cdot \lfloor\underline{r} \,.\, \&\text{c.}} \cdot a_0{}^p a_1{}^q a_2{}^r \,.\, \&\text{c.} \,.\, x^{q+2r+3s+\&\text{c.}};$$

and all the terms of the expansion may be found as before, by giving p, q, r, s, &c. all possible values which the condition $p + q + r + s + \&\text{c.} = m$ admits of.

Also any particular term involving a proposed power of x, as x^n, will be found by taking the sum of the values of this general term, when p, q, r, s, &c. are made to assume *all* the values, which satisfy the two equations $p + q + r + s + \&\text{c.} = m$, and $q + 2r + 3s + \&\text{c.} = n$.

Cor. 3. Assuming the Theorem for a positive integral index, it may be proved for a fractional or negative index thus:—

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[*] Σ stands for the expression "the sum of all the quantities of the form of."

[†] The proof here given of the Multinomial Theorem extends only to the case of positive integral indices, for by the Exponential Theorem m cannot be any thing but a positive integer. But if the Multinomial be deduced from the Binomial Theorem, (as in Art. 327) then since the latter is proved for fractional and negative indices, the former is also proved to hold for such indices.

Let $b+c+d+\&c.=x$, then $(a+b+c+d+\&c.)^m=(a+x)^m$, of which the general term, expressed by the $\overline{\alpha+1}^{\text{th}}$, is

$$\frac{m(m-1)\dots(p+1)}{1\,.\,2\,\dots\,\alpha}a^p x^\alpha,\ \text{ where } p+\alpha=m;$$

and since α is a positive integer, by what has been proved the general term of x^α, or $(b+c+d+\&c.)^\alpha$, is $\underline{|\alpha}\,.\,\frac{b^q}{\underline{|q}}\,.\,\frac{c^r}{\underline{|r}}\,.\,\&c.$

$$\therefore \text{ term required} = m(m-1)\dots(p+1)\,.\,a^p\,.\,\frac{b^q}{\underline{|q}}\,.\,\frac{c^r}{\underline{|r}}\,.\,\&c.$$

Ex. 1. Required the term in the expansion of $(a-b-c)^7$ which involves $a^2b^3c^2$.

Here $m=7$, $m-\pi=2$, $q=3$, $r=2$;

$$\therefore \text{ the term required } = \frac{7\,.\,6\,.\,5\,.\,4\,.\,3}{1\,.\,2\,.\,3\,.\,1\,.\,2}\,.\,a^2\,.\,(-b)^3\,.\,(-c)^2,$$

$$= -210a^2b^3c^2.$$

Ex. 2. Required the term in the expansion of $(a+bx+cx^2+dx^3)^4$ which involves x^8.

$$\text{The required term } = \Sigma\frac{\underline{|4}}{\underline{|p}\,.\,\underline{|q}\,.\,\underline{|r}\,.\,\underline{|s}}\,.\,a^p b^q c^r d^s\,.\,x^8$$

where $p+q+r+s=4$, and $q+2r+3s=8$; and it remains to find *all* the values of p, q, r, s which satisfy these equations. To do this, it is most convenient to take in order, beginning with the highest or the lowest, the several values of q, r, and s, which satisfy the latter, and reject those which are inconsistent with the former, equation. Thus beginning with q, we see that q cannot be greater than 4, therefore we try this value, which does not answer: then $q=3$, which also does not answer: then $q=2$, which answers with $p=0$, $r=0$, $s=2$; then $q=1$, which answers with $p=0$, $r=2$, $s=1$; then $q=0$, which answers with $p=1$, $r=1$, $s=2$. Next try $r=4$, which answers with $p=0$, $q=0$, $s=0$: then $r=3$, which does not answer; then $r=2$, $r=1$, $r=0$, which will not answer except as already determined. Next try $s=4$, 3, 2, 1, 0, in order; and no other values will answer except those already mentioned. The result, then, stands thus

p	q	r	s
0	2	0	2
0	1	2	1
1	0	1	2
0	0	4	0

where the required simultaneous values of p, q, r, s, are written from left to right. Hence the required term is

$$\underline{|4} \cdot \left\{ \frac{b^2d^2}{\underline{|2} \cdot \underline{|2}} + \frac{bc^2d}{\underline{|2}} + \frac{acd^2}{\underline{|2}} + \frac{c^4}{\underline{|4}} \right\} x^8$$

$$\text{or } (6b^2d^2 + 12bc^2d + 12acd^2 + c^4)x^8.$$

Ex. 3. Find the coefficient of x^8 in the expansion of $(a - bx + cx^2)^{12}$.

Here $\left.\begin{aligned} p+q+r &= 12 \\ q+2r &= 8 \end{aligned}\right\}$ are the equations of condition:

p	q	r
4	8	0
5	6	1
6	4	2
7	2	3
8	0	4

$$\therefore \text{ coefft. required} = \underline{|12} \cdot \left\{ \frac{a^4b^8}{\underline{|4} \cdot \underline{|8}} + \frac{a^5b^6c}{\underline{|5} \cdot \underline{|6}} + \frac{a^6b^4c^2}{\underline{|6} \cdot \underline{|4} \cdot \underline{|2}} + \frac{a^7b^2c^3}{\underline{|7} \cdot \underline{|2} \cdot \underline{|3}} + \frac{a^8c^4}{\underline{|8} \cdot \underline{|4}} \right\}$$

$$= 495a^4b^8 + 5544a^5b^6c + 13860a^6b^4c^2 + 7920a^7b^2c^3 + 495a^8c^4.$$

Ex. 4. Required the term involving x^{14} in the expansion of

$$(ax - bx^3 + cx^5 - \&c.)^{10}.$$

Here $(ax - bx^3 + cx^5 - \&c.)^{10}. = x^{10}(a - bx^2 + cx^4 - \&c.)^{10}$: therefore it will only be necessary to find the term involving x^4 in $(a - bx^2 + cx^4 - \&c.)^{10}$. Hence the equations of condition are

$$\left.\begin{aligned} p + q + r + \&c. &= 10 \\ 2q + 4r + \&c. &= 4 \end{aligned}\right\}$$

p	q	r
8	2	0
9	0	1

$$\therefore \text{ the term required} = \underline{|10} \cdot \left\{ \frac{a^8b^2}{\underline{|8} \cdot \underline{|2}} + \frac{a^9c}{\underline{|9}} \right\} x^{14}$$

$$= (45a^8b^2 + 10a^9c)x^{14}.$$

Ex. 5. Required the coefficient of x^3 in $(a+bx+cx^2+dx^3)^{\frac{1}{2}}$.

Here $\left.\begin{aligned} p+q+r+s &= \frac{1}{2} \\ q+2r+3s &= 3 \end{aligned}\right\}$ are the equations of condition:

p	q	r	s
$-\frac{5}{2}$	3	0	0
$-\frac{3}{2}$	1	1	0
$-\frac{1}{2}$	0	0	1

$$\therefore \text{ coefficient required } = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\frac{a^{-\frac{5}{2}}b^3}{\lfloor\underline{3}} + \frac{1}{2}\left(\frac{1}{2}-1\right)a^{-\frac{3}{2}}bc + \frac{1}{2}a^{-\frac{1}{2}}d$$

$$= \frac{b^3}{16a^{\frac{5}{2}}} - \frac{bc}{4a^{\frac{3}{2}}} + \frac{d}{2a^{\frac{1}{2}}}.$$

EVOLUTION OF SURDS.

A practical method of finding the square root of a binomial surd was ven in Art. 182; the following is the one more usually adopted:—

329. *To extract the square root of a quantity which is under the rm* $a+\sqrt{b}$.

$$\text{Assume } \sqrt{x}+\sqrt{y} = \sqrt{a+\sqrt{b}},$$

$$\text{then squaring, } x+y+2\sqrt{xy} = a+\sqrt{b};$$

$$\left.\begin{aligned} \therefore\ x+y &= a \\ \text{and } 2\sqrt{xy} &= \sqrt{b} \end{aligned}\right\} \text{(Art. 179)}:$$

om these two equations we find x and y, thus:—

$$\left.\begin{aligned} x^2+2xy+y^2 &= a^2 \\ 4xy &= b \end{aligned}\right\};$$

$$\therefore\ x^2-2xy+y^2 = a^2-b$$

$$x-y = \sqrt{a^2-b}.$$

$$\text{And } x+y = a;$$

$$\therefore\ 2x = a+\sqrt{a^2-b},$$

$$\text{and } 2y = a-\sqrt{a^2-b},$$

$$\therefore\ x = \frac{a+\sqrt{a^2-b}}{2},$$

$$\text{and } y = \frac{a - \sqrt{a^2 - b}}{2};$$

$$\therefore \sqrt{x} + \sqrt{y} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}.$$

From this conclusion it appears that the square root of $a + \sqrt{b}$ can only be expressed by a binomial, one or both of whose terms are *quadratic* surds, when $a^2 - b$ is a perfect square.

If the proposed surd be of the form $a - \sqrt{b}$, then we assume $\sqrt{x} - \sqrt{y} = \sqrt{a - \sqrt{b}}$, and proceed as before.

330. It must be observed that this method applies only to cases in which *one* of the terms of the binomial is a *quadratic* surd, and *the other rational.*

If, however, a binomial is proposed which can be put under the form $\sqrt{a^2c} + \sqrt{bc}$, or $\sqrt{c}\,(a + \sqrt{b})$, its square root may be found, by finding the square root of $a + \sqrt{b}$, and multiplying the result by $\sqrt[4]{c}$.

Ex. 1. Required the square root of $\frac{3}{2} + \sqrt{2}$.

$$\text{Assume } \sqrt{x} + \sqrt{y} = \sqrt{\frac{3}{2} + \sqrt{2}};$$

$$\text{then } x + y + 2\sqrt{xy} = \frac{3}{2} + \sqrt{2}$$

$$\left.\begin{aligned} \therefore\ x + y &= \frac{3}{2} \\ \text{and } 2\sqrt{xy} &= \sqrt{2} \end{aligned}\right\} \text{(Art. 179)}$$

$$\left.\begin{aligned} x^2 + 2xy + y^2 &= \frac{9}{4} \\ 4xy &= 2 \end{aligned}\right\}$$

$$x^2 - 2xy + y^2 = \frac{1}{4}$$

$$\left.\begin{aligned} x - y &= \frac{1}{2} \\ \text{and } x + y &= \frac{3}{2} \end{aligned}\right\};$$

$$\therefore 2x = 2, \text{ or } x = 1,$$

$$\text{and } 2y = 1, \text{ or } y = \frac{1}{2};$$

$$\therefore \sqrt{x} + \sqrt{y} = 1 + \frac{1}{2}\sqrt{2}, \text{ the root required.}$$

Ex. 2. Required the square root of $\sqrt{27}+\sqrt{24}$.

$$\text{Here } \sqrt{27}+\sqrt{24}=\sqrt{9\times 3}+\sqrt{8\times 3}=\sqrt{3}\,(3+\sqrt{8});$$

$$\therefore\ \sqrt{\sqrt{27}+\sqrt{24}}=\sqrt[4]{3}\,.\sqrt{3+\sqrt{8}};$$

nd applying the method of Art. 329, $\sqrt{3+\sqrt{8}}$ is found to be $1+\sqrt{2}$; (or e Art. 182. Ex. 1): therefore

the root required is $\sqrt[4]{3}\,(1+\sqrt{2})$, or $\sqrt[4]{3}+\sqrt[4]{12}$.

331. **Lemma.** *If* $\sqrt[3]{\mathrm{a}+\sqrt{\mathrm{b}}}=\mathrm{x}+\sqrt{\mathrm{y}}$, *then also* $\sqrt[3]{\mathrm{a}-\sqrt{\mathrm{b}}}=\mathrm{x}-\sqrt{\mathrm{y}}$.

For if $\sqrt[3]{a+\sqrt{b}}=x+\sqrt{y}$

$$a+\sqrt{b}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y},$$

$$\therefore \text{ (Art. 179) } a=x^3+3xy,$$

$$\text{and } \sqrt{b}=3x^2\sqrt{y}+y\sqrt{y}$$

$$\text{hence } a-\sqrt{b}=x^3-3x^2\sqrt{y}+3xy-y\sqrt{y},$$

$$\therefore\ \sqrt[3]{a-\sqrt{b}}=x-\sqrt{y}.$$

332. *To find the cube root of a binomial Surd of the form* $\mathrm{a}+\sqrt{\mathrm{b}}$, *hen it can be expressed by a binomial of the same description.*

$$\text{Assume } x+\sqrt{y}=\sqrt[3]{a+\sqrt{b}}$$

$$\text{then } x-\sqrt{y}=\sqrt[3]{a-\sqrt{b}}, \text{ (Art. 331)};$$

$$\therefore\ x^2-y=\sqrt[3]{a^2-b}.$$

Now, if a^2-b be a perfect cube, let it be equal to c^3;

$$\text{then } x^2-y=c$$

$$\text{but } x^3+3xy=a, \text{ (Art. 331)};$$

$$\therefore\ x^3+3x(x^2-c)=a,$$

$$\text{or } 4x^3-3cx=a.$$

From this equation x must be found by trial, and then y is known om the equation $y=x^2-c$; thus $x+\sqrt{y}$ is known, which is the required ot.

It appears from the operation that the cube root of $a+\sqrt{b}$ can only expressed by a binomial of the same form when a^2-b is a perfect cube. is test, therefore, ought to be applied to every proposed case in the st instance.

Ex. Required the cube root of $10+\sqrt{108}$.

Here $a^2-b=100-108=-8=(-2)^3$; therefore the method of Art. 2 may be applied.

$$\text{Let } x+\sqrt{y}=\sqrt[3]{10+\sqrt{108}}\,;$$
$$\therefore\ x-\sqrt{y}=\sqrt[3]{10-\sqrt{108}}\,;$$
$$x^2-y=\sqrt[3]{100-108}=-2.$$
$$\text{Also } x^3+3xy=10\,;$$
$$\therefore\ x^3+3x(x^2+2)=10,$$
$$\text{or } 4x^3+6x=10,$$

an equation which is satisfied by $x=1$; therefore $y=3$; and

$$x+\sqrt{y}=1+\sqrt{3}.$$

If therefore the cube root of $10+\sqrt{108}$ can be expressed in the proposed form, it is $1+\sqrt{3}$; which on trial is found to succeed.

333. **Lemma.** *If* n *be an odd number,* a *and* b *one or both quadratic surds, and* x and y *involve the same surds that* a *and* b *do respectively, and also* $(a+b)^{\frac{1}{n}}=x+y$, *then* $(a-b)^{\frac{1}{n}}=x-y$.

By involution $a+b=(x+y)^n$,

$$\text{or } a+b=x^n+nx^{n-1}y+n\frac{n-1}{2}x^{n-2}y^2+n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y^3+\&\text{c.}$$

where the odd terms involve the same surd that x does, because n is an odd number; and the even terms the same surd that y does; and since no part of a can consist of y, or its multiples or parts (Art. 181.),

$$a=x^n+n\frac{n-1}{2}x^{n-2}y^2+\&\text{c.}$$

$$\text{and } b=nx^{n-1}y+n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y^3+\&\text{c.},$$

hence,

$$a-b=x^n-nx^{n-1}y+n.\frac{n-1}{2}x^{n-2}y^2-n.\frac{n-1}{2}.\frac{n-2}{3}x^{n-3}y^3+\&\text{c.}$$
$$=(x-y)^n;$$
$$\therefore\ (a-b)^{\frac{1}{n}}=x-y.$$

334. *The* n$^{\text{th}}$ *root of a binomial, one or both of whose terms are possible quadratic surds, may sometimes be expressed by a binomial of that description.*

Let $A + B$ be the given binomial surd, in which both terms re possible; the quantities under the radical signs *whole numbers*; greater than B; and n an odd number.

$$\text{Assume } \sqrt[n]{(A + B) \times \sqrt{Q}} = x + y,$$

$$\text{then } \sqrt[n]{(A - B) \times \sqrt{Q}} = x - y, \text{ (Art. 333)}$$

$$\text{by mult. } \sqrt[n]{(A^2 - B^2) \times Q} = x^2 - y^2;$$

et Q be so assumed that $(A^2 - B^2) \times Q$ may be a perfect n^{th} power, s p^n, then $x^2 - y^2 = p$.

Again, by squaring both sides of the first two equations, we have

$$\sqrt[n]{(A + B)^2 \times Q} = x^2 + 2xy + y^2$$

$$\sqrt[n]{(A - B)^2 \times Q} = x^2 - 2xy + y^2$$

$$\text{hence } \sqrt[n]{(A + B)^2 \times Q} + \sqrt[n]{(A - B)^2 \times Q} = 2x^2 + 2y^2,$$

hich is always a whole number, when the root is a binomial of e supposed form; take therefore s and t the *nearest* integer lues of

$$\sqrt[n]{(A + B)^2 \times Q} \text{ and } \sqrt[n]{(A - B)^2 \times Q},$$

e of which is greater, and the other less than the true value of e corresponding quantity; then since the sum of these surds is integer, the fractional parts must destroy each other, and $x^2 + 2y^2 = s + t$ exactly, when the root of the proposed quantity n be obtained. We have therefore these two equations

$$\left.\begin{array}{l} x^2 - y^2 = p \\ x^2 + y^2 = \dfrac{s + t}{2} \end{array}\right\}$$

$$\therefore \quad 2x^2 = p + \frac{s + t}{2} = \frac{s + t + 2p}{2}$$

$$x^2 = \frac{s + t + 2p}{4}$$

$$x = \frac{\sqrt{s + t + 2p}}{2}.$$

$$\text{Also } 2y^2 = \frac{s+t-2p}{2}$$

$$\text{and } y = \frac{\sqrt{s+t-2p}}{2};$$

therefore, if the root of the binomial $\sqrt[n]{(A+B)\times\sqrt{Q}}$ be of the form $x+y$, it is $\frac{\sqrt{s+t+2p}+\sqrt{s+t-2p}}{2}$; and the n^{th} root of $A+B$ is $\frac{\sqrt{s+t+2p}+\sqrt{s+t-2p}}{2\sqrt[2n]{Q}}$.

335. In the same manner, the n^{th} root of $A-B$ is

$$\frac{\sqrt{s+t+2p}-\sqrt{s+t-2p}}{2\sqrt[2n]{Q}};$$

in which expression, when A is less than B, p is negative.

336. If the index of the root to be extracted be an *even* number, the square root of the proposed quantity may be found by Art. 329, when it can be expressed by a binomial of the same description; and if half the index be an even number, the square root may again be taken, and so on, until the root remaining to be extracted is expressed by an odd number, and then the method of the preceding Art., or of Art. 332, may be applied.

Ex. 1. Required the cube root of $11+5\sqrt{7}$.

Here $A=5\sqrt{7}$, $B=11$, $A^2-B^2=54$; therefore $Q=4$, and $p^3=216$, or $p=6$.

$$\begin{aligned}\text{Also } \sqrt[3]{(A+B)^2\times Q} &= \sqrt[3]{(296+110\sqrt{7})\times 4}\\ &= \sqrt[3]{2268{\cdot}44}\\ &= 13+f;\end{aligned}$$

$$\text{Similarly } \sqrt[3]{(A-B)^2\times Q} = 3-f;$$

or $s=13$, and $t=3$; therefore, by substitution, $x=\sqrt{7}$, and $y=1$; hence $x+y=\sqrt{7}+1$; and the quantity to be tried for the root is $\frac{\sqrt{7}+1}{\sqrt[3]{2}}$, which is found to succeed.

Ex. 2. Required the cube root of $2\sqrt{7} + 3\sqrt{3}$.

Here $A = 2\sqrt{7}$, $B = 3\sqrt{3}$, $A^2 - B^2 = 1$; hence $Q = 1$, and $= 1$.

$$\text{Also } \sqrt[3]{(A + B)^2 . Q} = \sqrt[3]{(55 + 12\sqrt{21})} \times 1$$
$$= \sqrt[3]{109{\cdot}96}$$
$$= 4 + f.$$

$$\text{Similarly } \sqrt[3]{(A - B)^2 . Q} = 1 - f;$$

$s = 4$, and $t = 1$; therefore $x = \frac{\sqrt{7}}{2}$, and $y = \frac{\sqrt{3}}{2}$; hence $x + y$ $\frac{\sqrt{7} + \sqrt{3}}{2}$, the quantity to be tried for the root, which is found succeed.

337. In the operation, it is required to find a number Q, such, at $(A^2 - B^2) \times Q$ may be a perfect n^{th} power; this will always be e case, if Q be taken equal to $(A^2 - B^2)^{n-1}$; but to find a less umber which will answer this condition, let $A^2 - B^2$ be divisible by a, a, &c....α times; b, b, &c....β times; c, c, &c....γ times, &c. succession; that is, let $A^2 - B^2 = a^\alpha b^\beta c^\gamma$, &c. Also let $Q = a^x b^y c^z$.&c. en

$$(A^2 - B^2) . Q = a^{\alpha + x} \times b^{\beta + y} \times c^{\gamma + z} \times \text{ \&c.}$$

hich is a perfect n^{th} power, if x, y, z, &c. be so assumed that $+ x$, $\beta + y$, $\gamma + z$, &c. are respectively equal to n, or some mulple of n. Thus, to find a number which multiplied by 180 will roduce a perfect cube, divide 180 as often as possible by 2, 3, 5, c. and it appears that $2 . 2 . 3 . 3 . 5 = 180$; if, therefore, it be mulplied by $2 . 3 . 5 . 5$, it becomes $2^3 . 3^3 . 5^3$, or $(2 . 3 . 5)^3$, which is a erfect cube.

338. If A and B be divided by their greatest common meare, either integer or quadratic surd, in all cases where the n^{th} ot can be obtained by this method, Q will either be unity, or some wer of 2, less than 2^n. See *Dr. Waring's Med. Alg.* Chap. v.

339. The square root of a *multinomial*, of which one term is rational, d the rest quadratic surds, may sometimes be found by assuming

$$\sqrt{a + \sqrt{b} + \sqrt{c} + \sqrt{d}} = \sqrt{x} + \sqrt{y} + \sqrt{z},$$

d proceeding to find x, y, and z, as in Art 329.

Ex. Required the square root of $21 + 6\sqrt{5} + 6\sqrt{7} + 2\sqrt{35}$.

Let $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{21 + 6\sqrt{5} + 6\sqrt{7} + 2\sqrt{35}}$,

then $x + y + z + 2\sqrt{xy} + 2\sqrt{xz} + 2\sqrt{yz} = 21 + 6\sqrt{5} + 6\sqrt{7} + 2\sqrt{35}$;

$$\therefore \left.\begin{array}{l} x + y + z = 21 \\ 2\sqrt{xy} = 6\sqrt{5} \\ 2\sqrt{xz} = 6\sqrt{7} \\ 2\sqrt{yz} = 2\sqrt{35} \end{array}\right\}, \text{ to find } x, y, \text{ and } z.$$

Now $2\sqrt{xy} \times 2\sqrt{xz} = 4x\sqrt{yz}$, or $6\sqrt{5} \times 6\sqrt{7} = 4x \cdot \sqrt{35}$,

$$\therefore x = 9, \text{ and } \sqrt{x} = 3.$$

Also $2\sqrt{xy} \times 2\sqrt{yz} = 4y\sqrt{xz}$, or $6\sqrt{5} \times 2\sqrt{35} = 12y\sqrt{7}$,

$$\therefore y = 5, \text{ and } \sqrt{y} = \sqrt{5}.$$

Again, $x + y + z = 21$, or $9 + 5 + z = 21$,

$$\therefore z = 7, \text{ and } \sqrt{z} = \sqrt{7}.$$

Hence $\sqrt{x} + \sqrt{y} + \sqrt{z} = 3 + \sqrt{5} + \sqrt{7}$, the root required.

INDETERMINATE COEFFICIENTS.

340. *If* $A + Bx + Cx^2 + \&c. = a + bx + cx^2 + \&c.$ *be an identical equation, that is, if it hold for all values whatever of* x, *then the coefficients of like powers of* x *are equal to each other, that is,* $A = a$, $B = b$, $C = c$, &c.

For if $A + Bx = a + bx$, then

$$A - a + (B - b)x = 0,$$

an equation which admits of one value of x only (Art. 193), unless $B - b = 0$, or $B = b$, and therefore also $A - a = 0$, or $A = a$.

Again, if $A + Bx + Cx^2 = a + bx + cx^2$, then

$$A - a + (B - b)x + (C - c)x^2 = 0,$$

a quadratic equation with respect to x which admits of no more than *two* distinct values of x (Art. 204), unless $C - c = 0$, or $C = c$, and $B - b = 0$, or $B = b$, and therefore also $A - a = 0$, or $A = a$.

Similarly, if *any number of terms* be taken, or

$$(A - a) + (B - b)x + (C - c)x^2 + \&c. = 0,$$

there are certain values of x, and none other, which will satisfy the equation as long as it remains an *equation* with respect to x.

But, by the supposition, the equation must be true for *any value whatever* which we may please to give to x, and consequently for *any* *number* of values of x; and this, therefore, can only be attained by that which is apparently an *equation* with respect to x ceasing to be such, that is, by the *coefficients* of the powers of x being separately equal to 0; that is, we must have

$$A - a = 0, \quad B - b = 0, \quad C - c = 0, \text{ \&c.}$$
$$\text{or } A = a, \quad B = b, \quad C = c, \text{ \&c.}$$

COR. If there be found any power of x on one side of the proposed equation, and no corresponding one on the other, then the whole coefficient of that power is of itself equal to 0. Thus, if $A + Bx + Cx^2 + \text{\&c.} = 0$, *for all values whatever of* x, then $A = 0$, $B = 0$, $C = 0$, &c.

341. It may be objected to the proof in the preceding Article, that it is rather assumed than proved that *every* equation has only a certain number of values of x which will satisfy it; and that it does not include those cases in which the assumed series is an "*infinite* series." No such objections can be made to the following proof[*]:—

Let $A + Bx + Cx^2 + \ldots = a + bx + cx^2 + \ldots$ be an identical equation, that is, hold for any value whatever of x; then

$$A \sim a + (B \sim b)\,x + (C \sim c)\,x^2 + \ldots = 0,$$

and, if $A \sim a$ is not equal to 0, let it be equal to some quantity p; then we have

$$(B \sim b)\,x + (C \sim c)\,x^2 + \text{\&c.} = p.$$

And since A, a, are invariable quantities, their difference p must be invariable; but $p = (B \sim b)\,x + (C \sim c)\,x^2 + \ldots$ a quantity which may have various values by the variation of x; that is, we have the same quantity (p) proved to be both fixed and variable, which is absurd. Therefore there is no quantity (p) which can express the difference $A \sim a$, or, in other words,

$$A \sim a = 0, \quad \text{and } \therefore A = a.$$

$$\text{Also } (B \sim b)\,x + (C \sim c)\,x^2 + \ldots = 0,$$
$$\text{or } B \sim b + (C \sim c)\,x + \ldots = 0. \quad \text{(Art. 82.)}$$

Therefore, by what has been proved, $B = b$. And so on, for the remaining coefficients of like powers of x.

342. *If*
$$\left.\begin{array}{r} A + Bx + Cx^2 + \ldots \\ + A'y + B'xy + \ldots \\ + A''y^2 + \ldots \end{array}\right\} = \left\{\begin{array}{r} a + bx + cx^2 + \ldots \\ + a'y + b'xy + \ldots \\ + a''y^2 + \ldots \end{array}\right.$$

---

[*] In the proof which is usually given x is assumed equal to 0, and afterwards the equal quantities are divided by x, whereas it is not proved that we may divide any quantity by x when x stands for 0, in the same manner as when it stands for a finite magnitude; and that such a proceeding will in certain cases lead to erroneous results is well known.

for all values whatever of x *and* y, *then the coefficients of like quantities are equal to each other, that is,*

$$A = a, \quad B = b, \quad C = c, \quad A' = a', \quad B' = b', \quad A'' = a'', \text{ \&c.}$$

Since x may receive any value whatever, suppose it to have some fixed value while y is variable, then the equation may be put under the form

$$A_{,} + B_{,}y + C_{,}y^2 + \ldots = a_{,} + b_{,}y + c_{,}y^2 + \ldots$$

where $A_{,}$, $B_{,}$, $C_{,}$, &c. $a_{,}$, $b_{,}$, $c_{,}$, &c. are invariable coefficients, and

$$\begin{aligned} A_{,} &= A + Bx + Cx^2 + \ldots \\ B_{,} &= A' + B'x + C'x^2 + \ldots \\ \text{\&c.} &= \text{\&c.} \\ a_{,} &= a + bx + cx^2 + \ldots \\ b_{,} &= a' + b'x + c'x^2 + \ldots \\ \text{\&c.} &= \text{\&c.} \end{aligned}$$

Now, by Art. 340, $A_{,} = a_{,}$, $B_{,} = b_{,}$, $C_{,} = c_{,}$, &c.; therefore

$$\begin{aligned} A + Bx + Cx^2 + \ldots &= a + bx + cx^2 + \ldots \\ A' + B'x + C'x^2 + \ldots &= a' + b'x + c'x^2 + \ldots \\ A'' + B''x + C''x^2 + \ldots &= a'' + b''x + c''x^2 + \ldots \\ \text{\&c.} &= \text{\&c.} \end{aligned}$$

Then by Art. 340 again, since x may have any value whatever,

$$A = a, \quad B = b, \quad C = c, \quad A' = a', \quad B' = b', \quad A'' = a'', \text{ \&c.}$$

The application of the preceding Theory will be shewn in the following Examples:—

Ex. 1. Expand $\dfrac{a - bx}{a + cx}$ to four terms; that is, divide $a - bx$ by $a + cx$.

Let $\dfrac{a - bx}{a + cx} = A + Bx + Cx^2 + Dx^3 + \ldots$; in which the coefficients A, B, C, D, &c. remain to be determined.

$$\begin{aligned} a - bx &= Aa + Bax + Cax^2 + Dax^3 + \ldots \\ &\qquad + Acx + Bcx^2 + Ccx^3 + \ldots \\ &= Aa + (Ba + Ac)x + (Ca + Bc)x^2 + (Da + Cc)x^3 + \ldots; \end{aligned}$$

and equating coefficients of like powers of x,

$$Aa = a, \quad \text{or } A = 1.$$

$$Ba + Ac = -b, \quad \therefore Ba = -(b + c), \quad \text{or } B = -\frac{b + c}{a};$$

$$Ca + Bc = 0, \quad \therefore Ca = \frac{b + c}{a} \cdot c, \quad \text{or } C = \frac{b + c}{a^2} \cdot c;$$

$$Da + Cc = 0, \quad \therefore Da = -\frac{b + c}{a^2} \cdot c^2, \quad \text{or } D = -\frac{b + c}{a^3} \cdot c^2;$$

$$\therefore\ \frac{a-bx}{a+cx}=1-\frac{b+c}{a}x+\frac{b+c}{a^2}cx-\frac{b+c}{a^3}c^2x^2+\ldots\ldots$$

Ex. 2. Extract the square root of $1+x$.

$$\text{Let } \sqrt{1+x}=1+Ax+Bx^2+Cx^3+\ldots\ldots$$

suming 1 for the first term, since that is the root when $x=0$;

$$\text{Squaring, } \left.\begin{aligned} 1+x=1+2Ax&+2Bx^2+2Cx^3+\ldots\ldots\ldots\ldots\\ &+A^2x^2+2ABx^3+\ldots\ldots\ldots\ldots\\ &\qquad\qquad +B^2x^4+\ldots \end{aligned}\right\}\text{(Art. 142).}$$

Hence, equating coefficients of like powers of x,

$$2A=1,\qquad \therefore\ A=\frac{1}{2};$$

$$2B+A^2=0,\qquad \therefore\ B=-\frac{A^2}{2}=-\frac{1}{8};$$

$$2C+2AB=0,\qquad \therefore\ C=-AB=+\frac{1}{16};\ \text{and so on;}$$

$$\therefore\ \sqrt{1+x}=1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\ldots\ldots\ldots$$

Ex. 3. Resolve $\dfrac{1}{(x+a)(x+b)(x+c)}$ into its partial fractions.

$$\text{Let } \frac{1}{(x+a)(x+b)(x+c)}=\frac{A}{x+a}+\frac{B}{x+b}+\frac{C}{x+c};$$

$$\therefore\ 1=A(x+b)(x+c)+B(x+a)(x+c)+C(x+a)(x+b).$$

Now, since this equation (by the theory) is true for *any* value of x,

let $x=-a$,

$$\text{then } 1=A(\overline{a-b}\,.\,\overline{a-c});\quad \text{or } A=\frac{1}{(a-b)(a-c)};$$

let $x=-b$,

$$\text{then } 1=-B(\overline{a-b}\,.\,\overline{b-c});\quad \text{or } B=-\frac{1}{(a-b)(b-c)};$$

let $x=-c$,

$$\text{then } 1=C(\overline{a-c}\,.\,\overline{b-c});\quad \text{or } C=\frac{1}{(a-c)(b-c)};$$

$$\frac{1}{(x+a)(x+b)(x+c)}=\frac{1}{(a-b)(a-c)(x+a)}-\frac{1}{(a-b)(b-c)(x+b)}$$
$$+\frac{1}{(a-c)(b-c)(x+c)}.$$

Ex. 4. Resolve $\dfrac{A + Bx + Cx^2}{(1 + ax)(1 + bx)(1 + cx)}$ into its partial fractions.

Let $\dfrac{A + Bx + Cx^2}{(1 + ax)(1 + bx)(1 + cx)} = \dfrac{P}{1 + ax} + \dfrac{Q}{1 + bx} + \dfrac{R}{1 + cx}$,

then $A + Bx + Cx^2 = P(1 + bx)(1 + cx) + Q(1 + ax)(1 + cx)$
$\qquad + R(1 + ax)(1 + bx)$,

let $x = -\dfrac{1}{a}$, that is, $1 + ax = 0$,

$$\text{then } A - \frac{B}{a} + \frac{C}{a^2} = P\left(1 - \frac{b}{a}\right)\left(1 - \frac{c}{a}\right); \text{ or } P = \frac{Aa^2 - Ba + C}{(a - b)(a - c)};$$

let $x = -\dfrac{1}{b}$, that is, $1 + bx = 0$,

$$\text{then } A - \frac{B}{b} + \frac{C}{b^2} = Q\left(1 - \frac{a}{b}\right)\left(1 - \frac{c}{b}\right); \text{ or } Q = -\frac{Ab^2 - Bb + C}{(a - b)(b - c)};$$

let $x = -\dfrac{1}{c}$, that is, $1 + cx = 0$,

$$\text{then } A - \frac{B}{c} + \frac{C}{c^2} = R\left(1 - \frac{a}{c}\right)\left(1 - \frac{b}{c}\right); \text{ or } R = \frac{Ac^2 - Bc + C}{(a - c)(b - c)};$$

$$\therefore \frac{A + Bx + Cx^2}{(1 + ax)(1 + bx)(1 + cx)} = \frac{Aa^2 - Ba + C}{(a - b)(a - c)(1 + ax)}$$
$$- \frac{Ab^2 - Bb + C}{(a - b)(b - c)(1 + bx)} + \frac{Ac^2 - Bc + C}{(a - c)(b - c)(1 + cx)}.$$

Ex. 5. Let $y^3 - 3y + x = 0$; required the value of y in a series of ascending powers of x.

Assume $y = Ax + Bx^3 + Cx^5 + Dx^7 + \&c.$*,

$$\left.\begin{array}{rl} \text{then } y^3 = & A^3x^3 + 3A^2Bx^5 + 3A^2Cx^7 + \&c. \\ & \qquad\qquad + 3AB^2x^7 + \&c. \\ -3y = & -3Ax - 3Bx^3 - 3Cx^5 - 3Dx^7 - \&c. \\ +x = & +x \end{array}\right\} = 0,$$

and supposing the coefficient of each power of x to be equal to 0, (Art. 340. Cor.)

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* The even powers of x are omitted because, from the given equation, it appears that the relation betwixt x and y is such that, if $-x$ be written for x, and $-y$ for y, the equation is not altered. If the even powers were retained, their coefficients would be found equal to 0.

$$-3A+1=0,\text{ or }A=\frac{1}{3};\quad A^3-3B=0,\quad\text{or }B=\frac{A^3}{3}=\frac{1}{3^4};$$

$$3A^2B-3C=0,\quad\text{or }C=A^2B=\frac{1}{3^6};\ \text{\&c.}$$

$$\therefore\ y=\frac{x}{3}+\frac{x^3}{3^4}+\frac{x^5}{3^6}+\text{\&c.}$$

Ex. 6. Let $x=ay+by^2+cy^3+$ &c. required the value of y in terms of x.

Assume $y=Ax+Bx^2+Cx^3+$ &c.

$$\left.\begin{array}{rlll}\text{then }ay= & aAx & +\,aBx^2+ & aCx^3+\text{\&c.}\\ by^2= & & bA^2x^2 & +\,2bABx^3+\text{\&c.}\\ cy^3= & & & cA^3x^3+\text{\&c.}\\ \text{\&c.}= & & & \text{\&c.}\\ -x= & -x & & \end{array}\right\}=0;$$

$$\text{hence }aA-1=0,\quad\text{or }A=\frac{1}{a};$$

$$aB+bA^2=0,\quad\text{or }B=\frac{-bA^2}{a}=\frac{-b}{a^3};$$

$$aC+2bAB+cA^3=0,\quad\text{or }C=\frac{-2bAB}{a}-\frac{cA^3}{a}=\frac{2b^2-ac}{a^5};\ \text{\&c.}$$

$$\therefore\ y=\frac{x}{a}-\frac{bx^2}{a^3}+\frac{(2b^2-ac)x^3}{a^5}+\text{\&c.}$$

Ex. 7. Let $x=y-ay^3+by^5-$ &c. required the value of y in terms of x.

Assume $y=Ax+Bx^3+Cx^5+$ &c.

$$\left.\begin{array}{rlll}\text{then }y= & Ax & +\,Bx^3 & +\,Cx^5+\text{\&c.}\\ -ay^3= & & -\,aA^3x^3 & -\,3aA^2Bx^5-\text{\&c.}\\ +by^5= & & + & bA^5x^5+\text{\&c.}\\ \text{\&c.}= & & & \text{\&c.}\\ -x= & -x & & \end{array}\right\}=0,$$

$$\text{hence }A-1=0,\ \text{or }A=1;\quad B-aA^3=0,\ \text{or }B=a;$$

$$C-3aA^2B+bA^5=0;\ \text{or }C=3a^2-b;\ \text{\&c.}$$

$$\therefore\ y=x+ax^3+(3a^2-b)x^5+\text{\&c.}$$

Ex. 8. *To find the sum of the series* $1^2+2^2+3^2+\ldots+n^2$ *by the method of Indeterminate Coefficients.*

Assume $1^2+2^2+3^2+\ldots+n^2=A+Bn+Cn^2+Dn^3+\&c.$ $A, B, C,$ &c. being unknown coefficients independent of n; then

$$1^2+2^2+\ldots+(n+1)^2=A+B(n+1)+C(n+1)^2+D(n+1)^3+\&c.$$

$\therefore$ by subtraction,

$$n^2+2n+1=B+2Cn+C+3Dn^2+3Dn+D,$$

the following terms being omitted, because their coefficients are separately equal to 0, being the coefficients in the last equation of $n^3, n^4,$ &c. Hence, equating coefficients of like powers of n (Art. 340),

$$\begin{aligned} 3D&=1, &\quad&\text{or } D=\tfrac{1}{3},\\ 2C+3D&=2, &&\text{or } C=\tfrac{1}{2},\\ B+C+D&=1, &&\text{or } B=\tfrac{1}{6},\\ &&&\text{and } A=0;\end{aligned}$$

$$\therefore \text{ sum required}=\frac{n}{6}+\frac{n^2}{2}+\frac{n^3}{3}$$

$$=\frac{1}{6}n(n+1)(2n+1).$$

Ex. 9. *To find the sum of the series* $1^3+2^3+3^3+\ldots+n^3$.

Assume $1^3+2^3+3^3+\ldots n^3=A+Bn+Cn^2+Dn^3+En^4+\&c.$

$$\therefore 1^3+2^3+\ldots+(n+1)^3=A+B(n+1)+C(n+1)^2+D(n+1)^3+E(n+1)^4+\ldots$$

and by subtraction,

$$n^3+3n^2+3n+1=B+2Cn+C+3Dn^2+3Dn+D+4En^3+6En^2+4En+E,$$

omitting the remaining terms for the reason assigned in the preceding Example. Hence

$$\begin{aligned} 4E&=1, &\quad&\text{or } E=\tfrac{1}{4},\\ 3D+6E&=3, &&\text{or } D=\tfrac{1}{2},\\ 2C+3D+4E&=3, &&\text{or } C=\tfrac{1}{4},\\ B+C+D+E&=1, &&\text{or } B=0,\\ &&&\text{and } A=0;\end{aligned}$$

$$\therefore \text{ sum required}=\frac{n^2}{4}+\frac{n^3}{2}+\frac{n^4}{4}=\tfrac{1}{4}n^2(n^2+2n+1)$$

$$=\left(\frac{n(n+1)}{1\,.\,2}\right)^2.$$

Cor. Since $1 + 2 + 3 \ldots + n = \dfrac{n \, . \, (n+1)}{1 \, . \, 2}$, therefore

$$1^3 + 2^3 + 3^3 + \ldots n^3 = (1 + 2 + 3 + \ldots + n)^2.$$

Ex. 10. *To solve the equation* $x^4 + 1 = 0$.

Assume $x^4 + 1 = (x^2 - mx + 1)\,(x^2 - nx + 1)$, then

$$x^4 + 1 = x^4 - (m+n)x^3 + (mn+2)x^2 - (m+n)x + 1\,;$$

and equating coefficients of like powers of x,

$$m + n = 0, \text{ and } mn + 2 = 0,$$

$$\therefore \; n = -m, \text{ and } m^2 = 2, \text{ or } m = \pm\sqrt{2}\,;$$

$$\therefore \; x^4 + 1 = (x^2 + \sqrt{2} \, . \, x + 1)\,(x^2 - \sqrt{2} \, . \, x + 1) = 0,$$

$$\text{or } x^2 + \sqrt{2} \, . \, x + 1 = 0, \text{ and } x^2 - \sqrt{2} \, . \, x + 1 = 0,$$

from which two quadratics we obtain the four roots,

$$x = \frac{-1 \pm \sqrt{-1}}{\sqrt{2}}, \text{ and } \frac{1 \pm \sqrt{-1}}{\sqrt{2}}.$$

Ex. 11. *To expand* a^x *in a series of powers of* x.

Since $a^x = (1 + \overline{a-1})^x = 1 + (\overline{a-1} - \tfrac{1}{2}\,\overline{a-1}|^2 + \ldots.)x + Bx^2 + Cx^3 + \ldots\ldots$ (See Art. 325), assume

$$a^x = 1 + Ax + Bx^2 + Cx^3 + \ldots\ldots\,;$$

$$\text{then } a^y = 1 + Ay + By^2 + Cy^3 + \ldots\ldots,$$

$$\text{and } a^{x+y} = 1 + A(x+y) + B(x+y)^2 + C(x+y)^3 + \ldots\;\ldots$$

Multiply the first two series together, and equate coefficients of y with the last, and the result will be

$$A + 2Bx + 3Cx^2 + 4Dx^3 + \ldots. = A + A^2x + ABx^2 + ACx^3 + \ldots.$$

from which, by equating coefficients of like powers of x, we have

$$A = A\,; \quad 2B = A^2, \quad \therefore \; B = \frac{A^2}{2}\,;$$

$$3C = AB, \;\therefore\; C = \frac{A^3}{1 \, . \, 2 \, . \, 3}\,; \quad 4D = AC, \quad \therefore \; D = \frac{A^4}{1 \, . \, 2 \, . \, 3 \, . \, 4}\,; \text{ and so on.}$$

$$\text{Hence } a^x = 1 + \frac{Ax}{1} + \frac{A^2x^2}{1 \, . \, 2} + \frac{A^3x^3}{1 \, . \, 2 \, . \, 3} + \ldots\ldots$$

CONTINUED FRACTIONS.

343. *To represent* $\frac{a}{b}$ *in a continued fraction*[*].

Let b be contained p times in a, with a remainder c; again, let c be contained q times in b, with a remainder d, and so on; then we have

$$\begin{array}{r} b)\,a\,(p \\ \overline{c)\,b\,(q} \\ \overline{d)\,c\,(r} \\ \overline{e\ \&c.} \end{array}$$

$$a = pb + c, \quad b = qc + d, \quad c = rd + e, \quad \&c.$$

or
$$\frac{a}{b} = p + \frac{c}{b} = p + \frac{c}{qc+d} = p + \cfrac{1}{q + \frac{d}{c}}$$

$$= p + \cfrac{1}{q + \frac{d}{rd+e}} = p + \cfrac{1}{q + \cfrac{1}{r + \frac{e}{d}}} \quad \&c.$$

that is,
$$\frac{a}{b} = p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s + \&c.}}}$$

344. Cor. 1. An approximation may thus be made to the value of a fraction whose numerator and denominator are in too high terms; and the farther the division is continued, the nearer will the approximation be to the true value.

345. Cor. 2. This approximation is alternately less and greater than the true value. Thus p is less than $\frac{a}{b}$; and $p + \frac{1}{q}$ is greater, because a part of the denominator of the fraction is omitted: $q + \frac{1}{r}$ is too great for the denominator, therefore $p + \cfrac{1}{q + \frac{1}{r}}$ is less than $\frac{a}{b}$; and so on.

Def. The quantities p, q, r, &c. are called the *Partial Quotients;*

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[*] Although a '*continued fraction*' (see Def. Art. 8) *may* be of the form $p + \cfrac{\alpha}{q + \cfrac{\beta}{r + \&c.}}$ the term is commonly restricted to those of the form $p + \cfrac{1}{q + \cfrac{1}{r + \&c.}}$, or $\cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r + \&c.}}}$

and $\frac{p}{1}$, $p+\frac{1}{q}$, $p+\cfrac{1}{q+\cfrac{1}{r}}$, when reduced to simple fractions, are called the *Converging Fractions*, or *Convergents*, to $\frac{a}{b}$.

Ex. To find a fraction which shall be nearly equal to $\frac{314159}{100000}$, and in lower terms.

```
100000) 314159 (3
        300000
        ------
         14159) 100000 (7
                 99113
                 -----
                   887) 14159 (15
                         887
                         ----
                         5289
                         4435
                         ----
                          854) 887 (1
                               854
                               ---
                                33 &c.
```

Here $p = 3$, $q = 7$, $r = 15$, $s = 1$, &c. therefore

$$\frac{314159}{100000} = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \&c.}}$$

The first approximation is 3, which is too little, the next is $3 + \frac{1}{7} = \frac{22}{7}$, too great; the next is $3 + \cfrac{1}{7 + \cfrac{1}{15}} = 3 + \frac{15}{106} = \frac{333}{106}$, too little; and so on.

The proposed fraction expresses nearly the circumference of a circle whose diameter is 1; therefore the circumference is greater than 3 diameters, less than $\frac{22}{7}$ diameters, and greater than $\frac{333}{106}$ diameters, &c.

346. *To convert any continued fraction into a series of converging fractions.*

Let the continued fraction be (Art. 343)

$$p + \cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{s + \&c.,}}}$$

then the converging fractions are

$$\frac{p}{1},\quad p+\frac{1}{q},\quad p+\cfrac{1}{q+\cfrac{1}{r}},\quad p+\cfrac{1}{q+\cfrac{1}{r+\cfrac{1}{s}}},\quad \&c.$$

$$\text{or } \frac{p}{1},\quad \frac{pq+1}{q},\quad \frac{pqr+p+r}{qr+1},\quad \frac{pqrs+ps+rs+pq+1}{qrs+q+s},\quad \&c.$$

$$\text{or } \frac{p}{1},\quad \frac{pq+1}{q},\quad \frac{(pq+1)r+p}{qr+1},\quad \frac{(\overline{pq+1}\,.\,r+p)s+pq+1}{(qr+1)s+q},\quad \&c.$$

in which the law of formation is observed to be as follows:—

Write down in one line the quotients p, q, r, s, &c., and the first and second converging fractions at sight; then the other fractions may be obtained thus:—

For the 3rd,

$$\begin{cases}\text{num}^{\text{r}}. = \text{3rd quot.} \times \text{num}^{\text{r}}.\text{ of 2nd fract.} + \text{num}^{\text{r}}.\text{ of 1st fract.}\\ \text{denom}^{\text{r}}. = \text{3rd quot.} \times \text{denom}^{\text{r}}.\text{ of 2nd fract.} + \text{denom}^{\text{r}}.\text{ of 1st fract.}\end{cases}$$

For the 4th,

$$\begin{cases}\text{num}^{\text{r}}. = \text{4th quot.} \times \text{num}^{\text{r}}.\text{ of 3rd fract.} + \text{num}^{\text{r}}.\text{ of 2nd fract.}\\ \text{denom}^{\text{r}}. = \text{4th quot.} \times \text{denom}^{\text{r}}.\text{ of 3rd fract.} + \text{denom}^{\text{r}}.\text{ of 2nd fract.}\end{cases}$$

And generally, for the n^{th} fraction in the series,

Multiply the n^{th} quotient by the numerator of the $\overline{n-1|}^{\text{th}}$ fraction and add the product to the numerator of the $\overline{n-2|}^{\text{th}}$ fraction. This will give the numerator.

Multiply the n^{th} quotient by the denominator of the $\overline{n-1|}^{\text{h}}$ fraction and add the product to the denominator of the $\overline{n-2|}^{\text{th}}$ fraction. This will give the denominator.

Ex. To find a series of converging fractions for $\frac{84}{227}$.

$$\frac{84}{227}=\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{2}}}}}}};$$

$\therefore$ the quotients are 2, 1, 2, 2, 1, 3, 2,

and the fractions are $\frac{1}{2}$, $\frac{1}{3}$, $\frac{3}{8}$, $\frac{7}{19}$, $\frac{10}{27}$, $\frac{37}{100}$, $\frac{84}{227}$.

347. *To express $\sqrt{a^2+1}$ in the form of a "Continued Fraction."*

$$\sqrt{a^2+1} = a + \sqrt{a^2+1} - a = a + \frac{1}{\sqrt{a^2+1}+a}$$

$$= a + \frac{1}{a + a + \sqrt{a^2+1} - a}$$

$$= a + \cfrac{1}{2a + \cfrac{1}{2a + \cfrac{1}{2a + \ldots\ldots}}}$$

a result which is easily remembered and applied to any proposed case.

Ex. $\sqrt{17} = \sqrt{4^2+1} = 4 + \cfrac{1}{8 + \cfrac{1}{8 + \cfrac{1}{8 + \ldots\ldots}}}$

The converging fractions will be $\frac{4}{1}$, $4 + \frac{1}{8}$, $4 + \cfrac{1}{8 + \frac{1}{8}}$, &c.

or $\frac{4}{1}$, $\frac{33}{8}$, $\frac{268}{65}$, &c. each of which is nearer to the true value of $\sqrt{17}$ than the one preceding.

348. *To express $\sqrt{n}$ in a continued fraction; and to find the converging fractions.*

Let a be the greatest whole number less than $\sqrt{n}$; then

$$\sqrt{n} = a + \sqrt{n} - a = a + \frac{n-a^2}{\sqrt{n}+a} = a + \cfrac{1}{\cfrac{\sqrt{n}+a}{r}}, \text{ where } r = n - a^2.$$

Let b be the greatest whole number less than $\frac{\sqrt{n}+a}{r}$; then

$$\frac{\sqrt{n}+a}{r} = b + \frac{\sqrt{n}+a-rb}{r} = b + \frac{\sqrt{n}-a'}{r}, \text{ if } a' = rb - a,$$

$$= b + \cfrac{1}{\cfrac{\sqrt{n}+a'}{r'}}, \text{ if } r' = \frac{n-a'^2}{r}.$$

Similarly $\frac{\sqrt{n}+a'}{r'} = b' + \cfrac{1}{\cfrac{\sqrt{n}+a''}{r''}}$, if $a'' = r'b' - a'$, and $r'' = \frac{n-a''^2}{r'}$;

and so on, until the quantity corresponding to r'' is equal to 1, after which the quotients will *recur*. The *quotients* of the continued fraction being thus found, the *converging fractions* will be found as in Art. 346.

Ex. $$\sqrt{11} = 3 + \sqrt{11} - 3 = 3 + \frac{2}{\sqrt{11}+3} = 3 + \frac{1}{\dfrac{\sqrt{11}+3}{2}},$$

$$\frac{\sqrt{11}+3}{2} = 3 + \frac{\sqrt{11}-3}{2} = 3 + \frac{2}{2(\sqrt{11}+3)} = 3 + \frac{1}{3+\sqrt{11}};$$

$$\therefore\ \sqrt{11} = 3 + \cfrac{1}{3 + \cfrac{1}{3+\sqrt{11}}} = 3 + \cfrac{1}{3 + \cfrac{1}{3+3+\cfrac{1}{3+\cfrac{1}{3+3+\cfrac{1}{3+\&c.}}}}}$$

$$= 3 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \cfrac{1}{3 + \cfrac{1}{6 + \&c.}}}}}}$$

the quotients are 3, 3, 6, 3, 6, &c.

the fractions are $\frac{3}{1}$, $\frac{10}{3}$, $\frac{63}{19}$, $\frac{199}{60}$, $\frac{1257}{379}$, &c.

349. As a test of the correctness of the converging fractions, it may be observed, that *the difference between any two consecutive fractions is always a fraction having* +1 *or* −1 *for its numerator,* according as that which is subtracted from the other is in an odd or even place. For

$$\frac{p}{1} - \frac{pq+1}{q} = \frac{-1}{q}, \quad \frac{pq+1}{q} - \frac{(pq+1)r+p}{qr+1} = \frac{1}{q(qr+1)};$$

$$\frac{(pq+1)r+p}{qr+1} - \frac{(\overline{pq+1}\cdot r+p)s+pq+1}{(qr+1)s+q} = \frac{-1}{(qr+1)(\overline{qr+1}\cdot s+q)};$$

and so on.

Thus, in the last example,

$$\frac{3}{1} - \frac{10}{3} = -\frac{1}{3}, \quad \frac{10}{3} - \frac{63}{19} = \frac{1}{57}, \quad \frac{63}{19} - \frac{199}{60} = -\frac{1}{1140}, \text{ \&c.}$$

Again, *every converging fraction,* correctly obtained according to the Rule, *is in its lowest terms;* for if $\frac{A}{B}$ and $\frac{C}{D}$ be any two consecutive ones, we know that $\frac{A}{B} - \frac{C}{D} = \pm\frac{1}{BD}$, that is, $AD - BC = \pm 1$. Now if A and B have a common measure greater than 1, it will measure AD and BC, and, erefore, $AD \sim BC$ or 1, which is impossible, since it is greater than 1. ence $\frac{A}{B}$ is in its lowest terms.

350. Hence also we may determine the limit of the error in taking y convergent for the true fraction $\frac{a}{b}$.

For if $\frac{A}{B}$ and $\frac{C}{D}$ be any two consecutive convergents, we know that $D - BC = \pm 1$, (Art. 349); and of $\frac{A}{B}$ and $\frac{C}{D}$ we know that $\frac{a}{b} <$ one, and $>$ e other, (Art. 345); therefore the difference between $\frac{a}{b}$ and either of them less than $\frac{A}{B} \sim \frac{C}{D} < \frac{AD \sim BC}{BD} < \frac{1}{BD}$.

Or, since $D > B$, à fortiori, $\frac{a}{b} \sim \frac{A}{B} < \frac{1}{B^2}$.

Thus in Ex. Art. 345, $\frac{22}{7}$ differs from the value of 3·14159 by a antity less than $\frac{1}{7 \times 106}$ or $\frac{1}{742}$; and $\frac{333}{106}$ differs from the true value a quantity less than $\frac{1}{(106)^2}$, or $\frac{1}{11236}$.

351. *To find the value of a continued fraction, when the otients* q, r, s, &c. *recur in any certain order.*

Ex. 1. Let

$$\cfrac{1}{q + \cfrac{1}{r + \cfrac{1}{q + \cfrac{1}{r + \text{\&c. in inf.}}}}} = x;$$

$$\text{then } \frac{1}{q + \frac{1}{r + x}} = x, \text{ or } \frac{r + x}{qr + qx + 1} = x;$$

$$\text{hence } r + x = qrx + qx^2 + x, \text{ and } x^2 + rx - \frac{r}{q} = 0,$$

y the solution of which quadratic equation the value of x may be btained.

Ex. 2. Let $x = \sqrt{a + \cfrac{b}{\sqrt{a + \cfrac{b}{\sqrt{a + \&c.}}}}}$ in inf.

squaring both sides, $x^2 = a + \cfrac{b}{\sqrt{a + \cfrac{b}{\sqrt{a + \&c.}}}} = a + \frac{b}{x}$;

and $x^3 - ax - b = 0$; whence the value of x is to be found.

352. The roots of quadratic equations may sometimes be approximated to by means of continued fractions. Thus,

Let $x^2 - px - q = 0$; then $x = p + \frac{q}{x} = p + \cfrac{q}{p + \cfrac{q}{x}}$, &c.;

so that $x = p + \cfrac{q}{p + \cfrac{q}{p + \cfrac{q}{p + \&c.}}}$, which, though not a contiuued fraction of such a form as those to which the preceding theory applies, may yet sometimes be made use of to calculate the value of x.

353. *To find in the form of a continued fraction the value of* x *which satisfies the equation* $a^x = b$.

Substitute for x the numbers 0, 1, 2, 3, &c. until two consecutive numbers are found, n, and $n + 1$, such that

$$a^n < b, \text{ and } a^{n+1} > b;$$

then it appears that $x < n + 1$, and $> n$; so that $x = n + \frac{1}{y}$, where $y > 1$, and $a^{n+\frac{1}{y}} = b$, or $a^n . a^{\frac{1}{y}} = b$, and $a^{\frac{1}{y}} = \frac{b}{a^n}$, or $\left(\frac{b}{a^n}\right)^y = a$. Again, since $\frac{b}{a^n} > 1$, and $< a$, by substituting the numbers between 1 and a for y in the last equation, two consecutive numbers, p, and $p + 1$, will be found such that $y < p + 1$, and $> p$, so that $y = p + \frac{1}{z}$; and $\therefore x = n + \cfrac{1}{p + \cfrac{1}{z}}$; and by continuing the process in the same manner, the fraction, expressing the value of x, may be continued.

Ex. Required the value of x in $10^x = 2$.

By substituting 0 and 1 for x, it appears that $x > 0$, and < 1; let $x = \frac{1}{y}$, then $10^{\frac{1}{y}} = 2$, or $2^y = 10$. Again, it appears that $y > 3$, and < 4; let $y = 3 + \frac{1}{z}$, then $2^{3+\frac{1}{z}} = 2^3 . 2^{\frac{1}{z}} = 10$, or $2^{\frac{1}{z}} = \frac{10}{8} = 1{\cdot}25$, $\therefore (1{\cdot}25)^z = 2$.

Again, it appears that $z > 3$, and < 4; let $z = 3 + \frac{1}{u}$, then $(1{\cdot}25)^{3+\frac{1}{u}} = (1{\cdot}25)^3.(1{\cdot}25)^{\frac{1}{u}} = 2$, $\therefore (1{\cdot}25)^{\frac{1}{u}} = \frac{2}{(1{\cdot}25)^3} = 1{\cdot}024$, $\therefore (1{\cdot}024)^u = 1{\cdot}25$, and by trial $u > 9$ and < 10. Hence by successive substitution

$$x = \cfrac{1}{3 + \cfrac{1}{3 + \cfrac{1}{9 + \&c.}}}$$

INDETERMINATE EQUATIONS AND UNLIMITED PROBLEMS.

354. WHEN there are more unknown quantities than independent equations, the number of corresponding values which those quantities admit is indefinite (Art. 198). This number may be lessened by rejecting all the values which are not integers; it may be farther lessened by rejecting all the negative values; and still farther, by rejecting all values which are not square or cube numbers; &c. By restrictions of this kind, the number of answers may be confined within definite limits; and problems are not wanting, in which such restrictions must be made.

355. *If a simple equation express the relation of two unknown quantities, and their corresponding* integral *values be required, divide the whole equation by the coefficient which is the less of the two, and suppose that part of the result, which is in a fractional form, equal to some whole number; thus a new simple equation is obtained, with which we may proceed as before; let the operation be repeated, till the coefficient of one of the unknown quantities is* 1, *and the coefficient of the other a whole number; then an integral value of the former may be obtained by substituting* 0, *or any whole number, for the other; and from the preceding equations integral values of the original unknown quantities may be found.*

Ex. 1. Let $5x + 7y = 29$; to find the corresponding integral values of x and y.

Dividing the whole equation by 5, the less coefficient,

$$x + y + \frac{2y}{5} = 5 + \frac{4}{5},$$

$$\text{or} \quad x = 5 - y + \frac{4-2y}{5}, \text{ a whole number;}$$

$$\therefore \frac{4-2y}{5} \text{ is a whole number.}$$

Assume $\dfrac{4-2y}{5} = p$, or $4 - 2y = 5p$,

$$\text{then} \quad 2 - y = 2p + \frac{p}{2};$$

$$\therefore y = 2 - 2p - \frac{p}{2}, \text{ a whole number.}$$

And $\dfrac{p}{2}$ is a whole number.

Let $\dfrac{p}{2} = s$, or $p = 2s$, then $y = 2 - 5s$,

$$\text{and} \quad x = 5 - y + p = 3 + 5s + 2s = 3 + 7s^*.$$

If $s = 0$, then $x = 3$ and $y = 2$, the only *positive* whole numbers which answer the conditions of the equation; for, if x and y are positive integers, $5s$ cannot be greater than 2, that is, s cannot be greater than $\frac{2}{5}$, and s cannot be negative, for then x would be negative.

If negative values of x and y are not excluded, then an indefinite number of such solutions may be found by putting 1, 2, 3, &c. $-1, -2$ -3, &c. for s.

Ex. 2. To find a number which being divided by 3, 4, 5, gives the remainders 2, 3, 4, respectively.

Let x be the number,

$$\text{then} \quad \frac{x-2}{3} = p, \text{ a whole number, or } x = 3p + 2;$$

also, from the second condition,

$$\frac{x-3}{4}, \text{ or } \frac{3p-1}{4} = q, \text{ a whole number,}$$

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[*] This operation might have been abridged thus,

$$x = 5 - y + \frac{2(2-y)}{5}; \text{ let } \frac{2-y}{5} = s, \text{ or } y = 2 - 5s,$$

$$\text{then } x = 5 - y + 2s = 3 + 7s.$$

that is, $3p - 1 = 4q$, or $p = q + \frac{q+1}{3}$;

let $\frac{q+1}{3} = r$, or $q = 3r - 1$, then $p = 4r - 1$,

and $x = 3p + 2 = 12r - 1$;

again, from the third condition, $\frac{x-4}{5}$ or $\frac{12r-5}{5}$ is a whole number, that is, $2r + \frac{2r}{5} - 1$ is a whole number;

$\therefore \frac{2r}{5}$ is a whole number;

let $\frac{2r}{5} = 2m$; then $r = 5m$,

and $x = 12r - 1 = 60m - 1$.

If $m = 1$, $x = 59$; if $m = 2$, $x = 119$; &c.

The artifices employed in the two following examples are deserving of notice.

Ex. 3. Let $11x - 17y = 5$; to find the integral values of x and y.

Here $x = \frac{17y+5}{11} = \frac{22y-5y+5}{11} = 2y - \frac{5(y-1)}{11}$ = a whole number.

Let $\frac{y-1}{11} = p$, or $y - 1 = 11p$; $\therefore y = 11p + 1$.

And $x = 2y - 5p = 22p + 2 - 5p = 17p + 2$.

If $p = 0$, $x = 2$, and $y = 1$; if $p = 1$, $x = 19$, and $y = 12$; &c.

Ex. 4. Let $11x - 18y = 63$; to find the integral values of x and y.

Since 18 and 63 are divisible by 9, let $x = 9z$, then

$11 \times 9z - 18y = 63$; and dividing by 9,

$11z - 2y = 7$;

$\therefore y = 5z - 3 + \frac{z-1}{2}$.

Let $\frac{z-1}{2} = p$, or $z - 1 = 2p$; $\therefore z = 2p + 1$;

Hence $x = 18p + 9$; and $y = 10p + 5 - 3 + p = 11p + 2$;

and by giving different integral values to p, all the values of x and y are determined.

356. If the simple equation contain more unknown quantities, their corresponding integral values may be found in the same manner.

Ex. Let $4x + 3y + 10 = 5z$; to find corresponding integral values of x, y and z.

Dividing the whole equation by 3, the least coefficient,

$$x + y + 3 + \frac{x+1}{3} = z + \frac{2z}{3},$$

$$y = z - x - 3 + \frac{2z - x - 1}{3}, \text{ a whole number.}$$

Assume $\frac{2z - x - 1}{3} = p$, or $2z - x - 1 = 3p$,

then $x = 2z - 3p - 1$,

and $y = z - 2z + 3p + 1 - 3 + p = 4p - z - 2$;

and for p and z substituting 0, or any whole numbers, integral values of x and y are obtained. If $z = 3$, and $p = 1$, then $x = 2$ and $y = -1$; if $z = 4$, and $p = 0$, then $x = 7$ and $y = -6$; &c.

357. If the number of independent equations be less by one than the number of unknown quantities, the equations may be reduced by elimination to one equation only betwixt two unknown quantities, and then the preceding method of solution may be applied.

Ex. Let $\left.\begin{aligned} 2x + 5y + 3z &= 108 \\ \text{and } 3x - 2y + 7z &= 95 \end{aligned}\right\}$; to find the integral values of x, y, z.

$$\left.\begin{aligned} \text{From 1st. equ}^{\text{n}}\quad 6x + 15y + 9z &= 324 \\ \ldots\ \text{2nd.}\ \ldots\quad 6x - 4y + 14z &= 190 \end{aligned}\right\};$$

$$\therefore\ 19y - 5z = 134;$$

from which equation the values of y and z may be found by the usual method, and then the values of x may be deduced from either of the original equations.

Indeterminate equations may also be solved as follows:—

358. *To find the integral values of* x *and* y *which satisfy an equation of the form* $ax + by = c$.

Suppose a and b prime to each other; for, if they are not, (since a, b, x, and y are all integers by supposition, and every common measure of a and b measures ax and by, and therefore $ax + by$, or c,) a, b, and

c must have a common measure, and dividing by this common measure, the equation is reduced to one in which the coefficients of x and y *are* prime to each other. Hence it is only necessary to consider the case when a and b are prime.

Let $x = \alpha$, $y = \beta$ be one solution; then

$$a\alpha + b\beta = c = ax + by,$$

$$\therefore\ a(\alpha - x) = b(y - \beta),$$

$$\text{and } \frac{a}{b} = \frac{y - \beta}{\alpha - x};$$

but $\frac{a}{b}$ is in its lowest terms, $\therefore\ y - \beta$, and $\alpha - x$, are certain equimultiples of a and b, or $y - \beta = at$, and $\alpha - x = bt$, where t is integral and remains undetermined;

$$\therefore\ x = \alpha - bt,$$

$$\text{and } y = \beta + at;$$

which values of x and y, upon substitution in the original equation, are found to satisfy the equation, whatever be the value of t, positive or negative.

Hence all the integral values of x and y which satisfy the proposed equation are found by giving different integral values to t in the equations

$$\left.\begin{aligned} x &= \alpha - bt \\ y &= \beta + at \end{aligned}\right\}.$$

The question, then, is reduced to finding *one* solution $x = \alpha$, $y = \beta$. This may be done by Art. 355, or more readily as follows:—

Find the series of fractions converging to $\frac{a}{b}$, and let $\frac{p}{q}$ be the fraction immediately preceding $\frac{a}{b}$, then (Art. 349) $\frac{a}{b} - \frac{p}{q} = \pm \frac{1}{bq}$, + or − according as $\frac{a}{b} >$ or $< \frac{p}{q}$;

$$\therefore\ aq - bp = \pm 1, \text{ and } a.cq - b.cp = \pm c;$$

and comparing this with the original equation, one solution is found to be

$$\left.\begin{aligned} x &= cq \\ y &= -cp \end{aligned}\right\}, \quad \text{or } \left.\begin{aligned} x &= -cq \\ y &= cp \end{aligned}\right\}, \text{ according as } \frac{a}{b} > \text{ or } < \frac{p}{q}.$$

Hence the general solution is either

$$\left.\begin{aligned} x &= cq - bt \\ y &= at - cp \end{aligned}\right\}, \quad \text{or } \left.\begin{aligned} x &= -cq - bt \\ y &= cp + at \end{aligned}\right\}.$$

COR. 1. If the equation be of the form $ax - by = c$, it is easily shewn that the general solution will be

$$\left.\begin{aligned} x &= \pm cq + bt \\ y &= \pm cp + at \end{aligned}\right\}, \ + \text{ or } - \text{ according as } \frac{a}{b} > \text{ or } < \frac{p}{q}.$$

Cor. 2. If an equation of the form $ax \pm by = c$ be proposed for solution, in which (when reduced as low as possible) *a and* b *are not prime to each other,* it may at once be affirmed that it admits of no solution in whole numbers. For, if x and y be whole numbers, dividing by the common measure of a and b, which is not a measure of c by supposition, we have a whole number equal to a fraction in lowest terms, which is impossible.

N. B. The converse proposition is not necessarily true, viz.—that, if a and b *are* prime to each other the equation has integral solutions.

Cor. 3. Since $x = \alpha - bt$, and $y = \beta + at$, is the general solution from which all the values of x and y are found by substituting for t, $0, \pm 1, \pm 2, \pm 3$, &c., it follows that the values of x and y, taken in order, form two arithmetic progressions, the one decreasing, and the other increasing; so that when two values of each are obtained, the rest may be assumed.

Ex. $5x + 7y = 29$; required the values of x and y in positive integers.

The converging fractions are $\frac{1}{1}, \frac{2}{3}, \frac{5}{7}$; and $\frac{p}{q} = \frac{2}{3}$, which $< \frac{5}{7}$;

$$\therefore\ x = 29 \times 3 = 87, \text{ and } y = -29 \times 2 = -58:$$

and the general solution is $\left.\begin{aligned} x &= 87 - 7t \\ y &= -58 + 5t \end{aligned}\right\}.$

Now in order that x and y may be both positive integers, t must be such, that $7t < 87$, and $5t > 58$, that is, $t < 12\frac{3}{7}$, and $> 11\frac{3}{5}$, or $t = 12$, and no other number.

Let $t = 12$, then $x = 3$, and $y = 2$: which is the only solution in positive integers.

359. *To find the* number *of solutions in positive integers of the equation* $ax + by = c$.

Let the series of fractions converging to $\frac{a}{b}$ be found, and let $\frac{p}{q}$ be that which immediately precedes $\frac{a}{b}$, then (Art. 349)

$$\text{either } aq - bp = 1, \text{ or } aq - bp = -1, \text{ according as } \frac{a}{b} > \text{ or } < \frac{p}{q}.$$

I. If $\frac{a}{b} > \frac{p}{q}$, then $a \cdot cq - b \cdot cp = c$,

$$\therefore\ a(cq - bt) + b(at - cp) = c;$$

and comparing this with the original equation,

$$x = cq - bt, \text{ and } y = at - cp;$$

and the several solutions will be found by giving t such values, that $cq - bt$ and $at - cp$ may be positive integral quantities, that is, t may be any positive integer less than $\frac{cq}{b}$ and greater than $\frac{cp}{a}$, but no other number, (*supposing* 0 *to be excluded as a value of either* x *or* y.)

Hence the number of solutions will be the number of integers which lie between $\frac{cq}{b}$ and $\frac{cp}{a}$.

II. If $\frac{a}{b} < \frac{p}{q}$, then $aq - bp = -1$, and it may be shewn, as before, that the number of solutions is the number of integers which lie between $\frac{cp}{a}$ and $\frac{cq}{b}$.

COR. 1. If $\frac{cp}{a}$, and $\frac{cq}{b}$ are both whole numbers, the number of solutions will be $\frac{cp}{a} \sim \frac{cq}{b} - 1$.

If one of the two be a whole number, and the other fractional, then the number of solutions will be the greatest integer contained in $\frac{cp}{a} \sim \frac{cq}{b}$.

If $\frac{cp}{a}$ and $\frac{cq}{b}$ be *both* fractional, reducing each to a mixed number, the number of solutions will be the greatest integer contained in $\frac{cp}{a} \sim \frac{cq}{b}$, or $\frac{cp}{a} \sim \frac{cq}{b} + 1$, according as the greater of the two quantities, $\frac{cp}{a}$, $\frac{cq}{b}$, has the greater or smaller fractional excess.

COR. 2. Since $\frac{cp}{a} \sim \frac{cq}{b} = c \, . \, \frac{bp \sim aq}{ab} = \frac{c}{ab}$, (Art. 349), the greatest integer contained in $\frac{c}{ab}$ cannot differ by more than 1 from the number of solutions; so that the number of solutions may be determined within this error (which may be either in excess or defect) without solving the equation at all.

COR. 3. If the given equation be of the form $ax - by = c$, $y = \frac{ax - c}{b}$, and therefore x may be any quantity greater than $\frac{c}{a}$ which is divisible by b, or the number of solutions is infinite.

Ex. In how many different ways may £1000. be paid in crowns and guineas?

Let x be the number of guineas, and y the number of crowns, then, reducing all to shillings,

$$21x + 5y = 20000;$$

and it is required to find how many solutions this equation admits of in positive integers.

$$\text{The fractions are } \frac{4}{1},\ \frac{21}{5};\ \therefore p = 4,\ q = 1;$$

$$\begin{aligned} \text{hence } \frac{cq}{b} \sim \frac{cp}{a} &= \frac{20000}{5} \sim \frac{20000 \times 4}{21}, \\ &= 4000 - 3809\tfrac{11}{21} \\ &= 190\tfrac{10}{21}; \end{aligned}$$

and since one of the quantities $\frac{cp}{a}$, $\frac{cq}{b}$, is a whole number,

$\therefore$ the number of solutions required is 190.

360. *To find the number of solutions in positive integers of the equation* $ax + by + cz = d$, *each term being positive**.

Transposing one of the terms, cz, the equation becomes

$$ax + by = d - cz,$$

where z may have any integral value greater than 0 and less than $\frac{d}{c}$; *zero* values of the unknown quantities being supposed to be excluded.

Hence, by Art. 359, the number of solutions will be the number of integers which lie between $\frac{(d-cz)p}{a}$, and $\frac{(d-cz)q}{b}$, where p and q are those quantities which satisfy the equation $ap - bq = 1$.

Thus a certain number of solutions is determined for each value of z, and the sum of these numbers will be the number required.

Obs. The application of this rule is attended with considerable difficulty (see *Barlow's Theory of Numbers*, Art. 162, or *Peacock's Algebra*, Art. 506); but the following Example will shew the student how he may determine the number of solutions in some cases without much trouble.

Ex. Required the number of integral solutions of

$$17x + 19y + 21z = 400.$$

$$\text{Dividing by 17,} \quad x + y + \frac{2y}{17} + z + \frac{4z}{17} = 23 + \frac{9}{17};$$

$$\therefore x + y + z = 23 - \frac{2y + 4z - 9}{17}.$$

$$\text{Assume } \frac{2y + 4z - 9}{17} = t, \quad \text{or} \quad 2y + 4z - 9 = 17t;$$

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* If any term be negative, as by, then the equation being of the form $ax - by = C$, the number of solutions is at once declared infinite. (Art. 359. Cor. 3).

then $y + 2z = 4 + 8t + \frac{t+1}{2}$.

Assume $\frac{t+1}{2} = s$; $\therefore\ t = 2s - 1$.

Hence it appears, (1) that t must be *an odd number;* (2) that it must be less than 23; also since $x + y + z$ cannot be less than 3, (*zero* values being excluded), t cannot be greater than 20, that is, its greatest value is 19; and therefore all the different values of t are

1, 3, 5, 7, 9, 11, 13, 15, 17, 19;

and the number of solutions is 10.

361. The theory for the solution of indeterminate equations of more than one *dimension* is too difficult to be admitted into an elementary work, like the present. The reader is referred for farther information to *Barlow's Theory of Numbers,* Chaps. III. and IV. But there are two classes of such equations, which admit of easy solution: 1st. Such as do not involve the square of *either* of the unknown quantities; and 2nd. Such as involve the square of *one* of them only.

Ex. 1. Required the integral solutions of $3xy - 4y + 3x = 14$.

$$\text{Here}\quad 3x(y+1) = 14 + 4y;$$

$$\therefore\ 3x = \frac{14+4y}{y+1} = \frac{4(y+1)}{y+1} + \frac{10}{y+1},$$
$$= 4 + \frac{10}{y+1};$$

hence $y + 1$ must be either 10, or a divisor of 10; that is, it must be either 1, 2, 5, or 10; and therefore the values of y can be only 0, or 1, or 4, or 9; of which the first and last make x negative.

$$\therefore\ \left.\begin{array}{l} x = 3,\ 2 \\ y = 1,\ 4 \end{array}\right\}\ \text{are the solutions required.}$$

Ex. 2. Required the integral solutions of $2xy - 3x^2 + y = 1$.

$$\text{Here}\quad y = \frac{3x^2+1}{2x+1} = \frac{3x}{2} - \frac{3}{4} + \frac{7}{4(2x+1)},\ \text{by division;}$$

$$\therefore\ 4y = 6x - 3 + \frac{7}{2x+1},\ \text{a whole number.}$$

Hence, 7 being a prime number, $2x + 1$ can be no other number but 7, or 1;

$$\therefore\ 2x = 6,\ \text{ or }\ x = 3,\ \text{ and }\ \therefore\ y = 4;$$
$$\text{or}\quad 2x = 0,\ \text{ i.e. }\ x = 0,\ \text{ and }\ \therefore\ y = 1.$$

Which are the only solutions in positive integers.

362. In the solution of different kinds of unlimited problems different expedients must be made use of, which expedients, and their application, are chiefly to be learned by practice.

Ex. 1. To find a "perfect number", that is, one which is equal to the sum of all the numbers which divide it without remainder.

Suppose $y^n x$ to be a "perfect number"; its divisors are

$$1,\ y,\ y^2 \dots y^n,\quad x,\ xy,\ xy^2 \dots xy^{n-1};$$

$$\therefore\ y^n x = 1 + y + y^2 \dots\dots + y^n + x + xy + xy^2 \dots\dots + xy^{n-1}.$$

Now $1 + y + y^2 \dots\dots + y^n = \dfrac{y^{n+1} - 1}{y - 1}$,

and $x + xy + xy^2 \dots\dots + xy^{n-1} = \dfrac{y^n - 1}{y - 1} \times x$;

$$\therefore\ y^n x = \frac{y^{n+1} - 1 + (y^n - 1) \times x}{y - 1},$$

or $y^{n+1}x - y^n x = y^{n+1} - 1 + y^n x - x$;

$$\therefore\ x = \frac{y^{n+1} - 1}{y^{n+1} - 2y^n + 1};$$

and, that x may be a whole number, let $y^{n+1} - 2y^n = 0$, or $y - 2 = 0$, that is, $y = 2$; then $x = 2^{n+1} - 1$.

Also, let n be so assumed that $2^{n+1} - 1$ may have no divisor but unity, which was supposed in taking the divisors of $y^n x$; then $y^n x$, or $2^n \times (2^{n+1} - 1)$ is a "perfect number." Thus, if $n = 1$, the number is 2×3 or 6, which is equal to $1 + 2 + 3$, the sum of its divisors: If $n = 2$, the number is $2^2 \times (2^3 - 1) = 4 \times 7 = 28$.

Ex. 2. To find two square numbers, whose sum is a square.

Let x^2 and y^2 be the two square numbers;

Assume $x^2 + y^2 = (nx - y)^2 = n^2x^2 - 2nxy + y^2$,

then $x^2 = n^2x^2 - 2nxy$,

$x = n^2x - 2ny$;

hence $(n^2 - 1)\,x = 2ny$,

or $x = \dfrac{2ny}{n^2 - 1}$.

And if n and y be assumed at pleasure, such a value of x is btained, that $x^2 + y^2$ is a square number.

But if it be required to find *integers* of this description, let $= n^2 - 1$, then $x = 2n$, and n being taken at pleasure, integral alues of x and y, and consequently of x^2 and y^2, will be found. hus, if $n = 2$, then $y = 3$, and $x = 4$, and the two squares are 9 nd 16, whose sum is 25, a square number.

Ex. 3. To find two square numbers, whose difference is a square.

Let x^2 and y^2 be the two squares;

$$\text{Assume } x^2 - y^2 = (x - ny)^2$$

$$= x^2 - 2nxy + n^2y^2.$$

$$\text{Then } y^2 = 2nxy - n^2y^2$$

$$\text{or } 2nx = (n^2 + 1)y;$$

$$\therefore\ x = \frac{n^2 + 1}{2n} \cdot y.$$

nd if $y = 2n$, then $x = n^2 + 1$. Thus, if $n = 2$, then $y = 4$, and $= 5$; hence $x^2 - y^2 = 25 - 16 = 9$.

SCALES OF NOTATION.

363. *To explain the different systems of notation.*

DEF. In the common system of notation each figure of any number[*] creases its value in a tenfold proportion in proceeding from right to left. hus 3256 may be expressed by

$$6 + 50 + 200 + 3000,$$

$$\text{or } 6 + 5 \times 10 + 2 \times 10^2 + 3 \times 10^3.$$

The figures 3, 2, 5, 6, by which the number is formed, are called its *gits*, and the number 10, according to whose powers their values proceed, called the *radix* of the scale.

It is purely conventional that 10 should be the *radix;* and therefore ere may be any number of different *scales*, each of which has its own *dix*. When the *radix* is 2, the *scale* is called *Binary;* when 3, *Ternary;* hen 10, *Denary*, or *Decimal;* when 12, *Duodenary*, or *Duodecimal;* &c.

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[*] In this Section and in the following one by *number* a *whole number* is always meant.

If 3256 expressed a number in a scale whose radix is 7, that number might be expressed thus,

$$6 + 5 \times 7 + 2 \times 7^2 + 3 \times 7^3.$$

And generally, if the digits of a number be a_0, a_1, a_2, a_3, &c., reckoning from right to left, and the radix r, the number will be properly represented by

$$a_0 + a_1 r + a_2 r^2 + a\ r^3 + \&c.$$

Or, if there be n digits, the number will be (reversing the order of the terms)

$$a_{n-1} r^{n-1} + a_{n-2} r^{n-2} + a_{n-3} r^{n-3} + \ldots + a_1 r + a_0.$$

Obs. In any scale of notation every digit is necessarily less than r, and the number of them, including 0, is equal to r. Also in any number the highest power of r is less by 1 than the number of digits.

364. *To express a given number in any proposed scale.*

Let N be the number, and r the radix of the proposed scale.

Then if a_0, a_1, a_2, &c. be the unknown digits,

$$N = a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \&c.;$$

and if N be divided by r, the remainder is a_0.

If the quotient be again divided by r, the remainder is a_1.

If .... is a_2,

and so on, until there is no further quotient.

Therefore all the digits a_0, a_1, a_2, a_3, &c. are found by these repeated divisions; and consequently the number in the proposed scale.

Ex. To express 1820, written according to the denary scale, in a scale whose radix is 6.

$$\begin{array}{r|r l}
6 & 1820 & \\ \cline{2-2}
6 & 303,\ 2 & \therefore\ \text{1st remainder } a_0 = 2, \\ \cline{2-2}
6 & 50,\ 3 & \text{2d remainder } a_1 = 3, \\ \cline{2-2}
6 & 8,\ 2 & \text{3d remainder } a_2 = 2, \\ \cline{2-2}
6 & 1,\ 2 & \text{4th remainder } a_3 = 2, \\ \cline{2-2}
\multicolumn{1}{r}{} & 0,\ 1 & \text{5th remainder } a_4 = 1;
\end{array}$$

$\therefore$ the number required is 12232.

This may be easily verified. Thus if the result be correct,

$$2 + 3 \times 6 + 2 \times 6^2 + 2 \times 6^3 + 1 \times 6^4$$

must amount to 1820; which, upon trial, it is found to do.

By the same method a number may be transformed from *any* given scale to any other of which the radix is given. It is only necessary to bear n mind throughout the process that the radix is not 10, as usual, but some other number. Or the same thing may be done by first expressing the given number in the *denary* scale, and then proceeding as in the last example.

Ex. Transform 3256 from a scale whose radix is 7 to a scale whose radix is 12.

Bearing in mind that the digits in 3256 increase from right to left in a *sevenfold* proportion, the division by twelve will be performed thus,

$$\begin{array}{r|l}
12 & 3256 \\ \hline
12 & 166,\ 4 \quad \therefore \text{ 1st remainder } a_0 = 4, \\ \hline
12 & 11,\ 1 \quad\ \ \text{ 2d remainder } a_1 = 1, \\ \hline
 & \ \ 0,\ 8 \quad\ \ \text{ 3d remainder } a_2 = 8;
\end{array}$$

∴ the number required is 814.

365. The most useful scale of notation, after the common denary scale, is the one which has *twelve* for its radix, called Duodecimal. Here it is necessary to have two new symbols, in addition to 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, for the purpose of expressing *ten* and *eleven;* since 10, and 11, in the duodenary scale, signify twelve and thirteen respectively. These new symbols may be any thing, distinct from the other digits, and are usually two letters, as *t* for ten, and *e* for eleven.

The common pence-table will be found of considerable service in the use of the duodecimal notation.

Ex. 1. Multiply 25 ft. 7 in. by 7 ft. 10 in.

The lengths are here expressed in the *denary* scale. In the duodenary the question is, what is the product of 21·7 by 7·*t*?

$$\begin{array}{r}
21{\cdot}7 \\
7{\cdot}t \\ \hline
193t \\
12e1\ \ \\ \hline
148{\cdot}4t
\end{array}$$

∴ the product is 148 ft. $4'.\,t''$ in the duodenary notation,

or $8 + 4 \times 12 + 1 \times 12^2$ ft. $4'.\ 10''$ in the denary;

that is, 200 ft. $4'.\ 10''$.....................

Ex. 2. A floor in the form of a rectangular parallelogram contains 1532 square feet, $9'.\,9''$, and is 81 ft. $9'$ long; required the breadth.

The whole area ÷ the length = the breadth; and in the duodenary scale the given quantities are *t*78·99 and 69·9 respectively.

```
69·9) t78·99 (16·9 ft. in the duodenary scale,
      669
      ----
      39e9
      34t6
      -----
       5139
       5139
       ----
       *  *
```

∴ in the common scale the breadth of the floor is 18 ft. 9 in.

366. *To find the greatest and least numbers with a given number of digits in any proposed scale.*

Let r be the radix of the scale, and n the number of digits; then it is evident that the number will be *greatest*, when every digit is as great as it can be, that is, $r-1$; in which case the number will be

$$r-1+(r-1)r+(r-1)r^2+\dots\dots+(r-1)r^{n-1},$$

$$\text{or } (r-1)(1+r+r^2+\dots\dots+r^{n-1});$$

$$\text{or } (r-1)\frac{r^n-1}{r-1}, \text{ or } r^n-1.$$

Again, the number will be *least*, when the extreme digit to the left is 1, and every other is 0; in which case it will be equal to r^{n-1}.

Ex. Thus in the common, or denary, scale, the greatest number having 4 digits is 10^4-1, or 9999; and the least is 10^3, or 1000.

367. *To find the number of digits in the product or quotient of two given numbers.*

I. Let P and Q represent the numbers, having p and q digits respectively; and r the radix of the scale; then

$$P=a_0+a_1r+\dots\dots+a_{p-1}r^{p-1},$$

$$Q=b_0+b_1r+\dots\dots+b_{q-1}r^{q-1};$$

∴ PQ = a series of terms in which the highest power of r is found in $a_{p-1}b_{q-1}r^{p+q-2}$, the greatest value of which is $(r-1)^2 . r^{p+q-2}$, and the least r^{p+q-2}. Therefore the highest power of r in PQ is less than $r^2 . r^{p+q-2}$, or r^{p+q}, and the lowest power is r^{p+q-2}. Hence the number of digits in PQ cannot be greater than $p+q$, nor less than $p+q-1$.

II. To find the number in $P \div Q$, when P is exactly divisible by Q; let $P \div Q = R$, then $P=QR$, and ∴ QR contains p digits. Suppose R to contain x digits; then, by former case, p cannot be greater than $q+x$, nor less than $q+x-1$; ∴ x, the number required, cannot be less than $p-q$, nor greater than $p-q+1$.

Cor. Hence the number of digits in P^2 is either $2p$ or $2p-1$; in P^3 either $3p$, or $3p-1$, or $3p-2$; and so on.

368. *In every system of notation, of which the radix is* r, *the sum of the digits of any number divided by* $\overline{r-1}$ *will leave the same remainder as the whole number divided by* $\overline{r-1}$.

Let $a_0 + a_1 r + a_2 r^2 + a_3 r^3 + \ldots\ldots + a_n r^n$ be the number (N),
then $N = a_0 + a_1 + a_2 + \ldots + a_n + a_1(r-1) + a_2(r^2-1) + \ldots\ldots + a_n(r^n - 1)$;

$$\therefore \frac{N}{r-1} = \frac{a_0 + a_1 + \ldots a_n}{r-1} + a_1 + a_2(r+1) + \ldots\ldots + a_n \cdot \frac{r^n - 1}{r-1};$$

But $\dfrac{r^n-1}{r-1}$ is a whole number, whatever positive whole number n may be; (Art. 99, Ex. 6);

$$\therefore \frac{N}{r-1} = P + \frac{a_0 + a_1 + \ldots + a_n}{r-1},\ P \text{ being a whole number,}$$

or the number divided by $\overline{r-1}$ leaves the same remainder as the sum of the digits divided by $\overline{r-1}$.

Cor. Hence, if the sum of the digits of any number be divisible by $\overline{r-1}$, the number itself is divisible by $\overline{r-1}$.

Similarly, it may be shewn that, if the difference between the sum of the digits in the odd places and the sum of the digits in the even places be divisible by $\overline{r+1}$, the number is divisible by $\overline{r+1}$.

Prob. *To find what numbers are divisible by 3 and 9 without remainders.*

Let a, b, c, d, &c. be the digits, or figures in the units', tens', hundreds', thousands', &c. place of any number, then the number is

$$a + 10b + 100c + 1000d + \&c.$$

this divided by 3 is

$$\frac{a}{3} + 3b + \frac{b}{3} + 33c + \frac{c}{3} + 333d + \frac{d}{3} + \&c.$$

$$\text{or } \frac{a+b+c+d+\&c.}{3} + 3b + 33c + 333d + \&c.$$

which is a whole number when $\dfrac{a+b+c+d+\&c.}{3}$ is a whole number; that is, any number is a multiple of 3 if the sum of its digits be a multiple of 3. Thus 111, 252, 7851, &c. are multiples of 3.

In the same manner, any number is a multiple of 9 if the sum of its digits be a multiple of 9.

$$\text{For } \frac{a + 10b + 100c + 1000d + \&c.}{9}$$

$$= \frac{a}{9} + b + \frac{b}{9} + 11c + \frac{c}{9} + 111d + \frac{d}{9} + \&c.$$

$$= \frac{a + b + c + d + \&c.}{9} + b + 11c + 111d + \&c.$$

which is a whole number when $\frac{a + b + c + d + \&c.}{9}$ is a whole number. Thus 684, 6588, &c. are multiples of 9.

Cor. 1. Hence, if any number, and the sum of its digits be respectively divided by 9, the remainders are equal.

Cor. 2. From this property of 9 may be deduced a rule which will *sometimes* detect an error in the multiplication of two numbers, called *casting out the Nines*.

Let the multiplicand be $9a + x$, that is, let it consist of a nines, with a remainder x; and let $9b + y$ be the multiplier, then

$81ab + 9bx + 9ay + xy$ is the product;

and if the sum of the digits in the multiplicand be divided by 9, the remainder is x; if the sum of the digits in the multiplier be divided by 9, the remainder is y; and if the sum of the digits in the product be divided by 9, the remainder is the same as when the sum of the digits in xy is divided by 9, if there be no mistake in the operation *.

PROPERTIES OF NUMBERS.

369. *The product of any two consecutive numbers is divisible by* 1×2.

Of the two numbers one must evidently be even, that is, divisible by 2, therefore their product is divisible by 2.

370. *The product of any three consecutive numbers is divisible by* $1.2.3$, *or* 6.

Every number must be either of the form $3m$, or $3m \pm 1$, or $3m \pm 2$, since it must be either divisible by 3 without remainder, or have a remainder 1 or 2; therefore the product of any three consecutive numbers may be represented by one of the forms

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* It will be easily seen that this method fails to detect an error in any of the following cases:—(1) when one or more cyphers have been omitted in the product; (2) when any of its digits are misplaced; and (3) when the error is equal to 9 or any multiple of 9.

$$3m(3m+1)(3m+2),$$
$$(3m-1)\,3m\,(3m+1),$$
$$(3m-2)(3m-1)\,3m.$$

Now, since by last Art. both $(3m+1)(3m+2)$ and $(3m-2)(3m-1)$ are divisible by 1.2, and $3m$ is a multiple of 3, therefore the first and third forms are divisible by $1.2.3$. Also, if m be an even number, that is, divisible by 2, it is clear that $3m$, and therefore the second form is divisible by $1.2.3$; and, if m be an odd number, $3m$ is odd, and $3m+1$ even; consequently the second form is divisible by $1.2.3$.

Hence, in all cases, the product of three consecutive numbers is divisible by $1.2.3$.

And generally,

371. *The continued product of any* r *consecutive numbers is divisible by* 1.2.3 ... r.

Let $f(m,r)$ represent $m.\dfrac{m-1}{2}.\dfrac{m-2}{3}\,\cdots\cdots\,\dfrac{m-r+1}{r}$,

$$\text{then } f(m, r+1) = m.\frac{m-1}{2}\cdots\cdots\frac{m-r+1}{r}.\frac{m-r}{r+1},$$

$$=f(m,r).\frac{m-r}{r+1}$$

$$=\frac{m+1}{r+1}f(m,r)-f(m,r)$$

$$=\frac{(m+1)\,m\,(m-1)\cdots\cdots(\overline{m+1}-\overline{r+1}+1)}{1\;.\;2\;.\;3\quad\cdots\cdots\quad(r+1)}-f(m,r)$$

$$=f(m+1,\,r+1)-f(m,r);$$

$$\therefore\; f(m+1,\,r+1)-f(m,r+1)=f(m,r).$$

$$\begin{aligned}
\text{Hence }\; & f(r+1,\,r+1) && =1,\\
& f(r+2,\,r+1)-f(r+1,\,r+1) && =f(r+1,\,r),\\
& f(r+3,\,r+1)-f(r+2,\,r+1) && =f(r+2,\,r),\\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots && =\cdots\cdots\cdots\\
& f(m,\,r+1)-f(m-1,\,r+1) && =f(m-1,\,r);
\end{aligned}$$

$$\therefore\; f(m,r+1)=1+f(r+1,\,r)+f(r+2,\,r)+\ldots+f(m-1,\,r).$$

If $r=1$, $f(m,2)=1+f(2,1)+f(3,1)+\cdots\cdots+f(m-1,1)$

$$=1+2+3+\cdots\cdots+\overline{m-1} \quad = \text{a whole number.}$$

If $r=2$,

$f(m,3)=1+f(3,2)+f(4,2)+\cdots\cdots+f(m-1,2)=$ a whole number.

If $r=3$,

$f(m,4)=1+f(4,3)+f(5,3)+\cdots\cdots+f(m-1,3)=$ a whole number.

&c. = &c. &c. = ...

And $f(m, r) = 1 + f(r, r-1) + f(r+1, r) + \dots + f(m-1, r-1)$ = a whole number; or the product of r consecutive numbers is divisible by $\underline{|r}$[*].

Ex. 1. If n be any whole number, then will $n(n^2-1)(n^2-4)$ be divisible by 120.

$$\begin{aligned} n(n^2-1)(n^2-4) &= n(n-1)(n+1)(n-2)(n+2) \\ &= (n-2)(n-1)n(n+1)(n+2), \end{aligned}$$

which is the product of 5 consecutive numbers, and is therefore divisible by $1.2.3.4.5$, or 120.

Ex. 2. If n be any even number, $n^3 + 20n$ is divisible by 48.

Let $n = 2m$, since it is an even number,

$$\begin{aligned} \text{then } n^3 + 20n &= 8m^3 + 40m \\ &= 8m(m^2+5) \\ &= 8m(m^2-1) + 48m \\ &= 8(m-1)m(m+1) + 48m. \end{aligned}$$

Now $(m-1)m(m+1)$ being the product of three consecutive numbers is divisible by $1.2.3$ or 6; therefore $n^3 + 20n$ is divisible by 48.

372. *Every number which is a perfect square is of one of the forms* $5m$ *or* $5m \pm 1$.

For every number is of one of the forms $5m$, $5m+1$, $5m+2$, $5m+3$, $5m+4$; all of which are included in the forms $5m$, $5m \pm 1$, $5m \pm 2$, since $5m+3 = 5(m+1) - 2 = 5m' - 2$, and $5m+4 = 5(m+1) - 1 = 5m' - 1$.

But $(5m)^2 = 5(5m^2) = 5m'$, which is of the form $5m$;

$(5m \pm 1)^2 = 25m^2 \pm 10m + 1 = 5(5m^2 \pm 2m) + 1$, which is of the form $5m+1$;

$(5m \pm 2)^2 = 25m^2 \pm 20m + 4 = 5(5m^2 \pm 4m + 1) - 1$, which is of the form $5m-1$;

$\therefore$ every square is of one of the forms $5m$, $5m+1$, $5m-1$.

373. *Every "prime number" greater than 2 is of one of the forms* $4m \pm 1$.

For every number is of one of the forms $4m$, $4m+1$, $4m+2$, $4m+3$; but neither $4m$, nor $4m+2$ can represent prime numbers, since each is divisible by 2; therefore all prime numbers greater than 2 are represented by $4m+1$ and $4m+3$. But $4m+3 = 4(m+1) - 1 = 4m' - 1$; therefore the two forms are $4m \pm 1$.

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[*] This proof is not so simple as that found in Barlow's *Theory of Numbers*, Art. 12, and usually adopted; but the latter is liable to serious objection; for it *assumes* that, if the numerator of $\frac{n(n-1)(n-2)\,\&c.}{1\;.\;2\;.\;3\;.\;\&c.}$ be divisible by each factor of the denominator *separately*, it is divisible also by their continued product: which requires proof, since a number may obviously be divisible by each of two or more numbers, and yet not be divisible by their product.

Cor. Since m may be odd or even, that is, of the form $2n$, or $2n+1$, all prime numbers are represented by $8n \pm 1$, or $8n \pm 3$.

374. *Every prime number greater than 3 is of one of the forms* $6m \pm 1$.

For every number is of one of the forms $6m$, $6m+1$, $6m+2$, $6m+3$, $6m+4$, $6m+5$, of which the 1st, 3d, 4th, and 5th obviously cannot represent *prime* numbers; and therefore, all prime numbers greater than 3 are represented by $6m+1$ and $6m+5$. But $6m+5 = 6(m+1)-1 = 6m'-1$. Therefore, $6m \pm 1$ will include all prime numbers greater than 3.

Cor. Since m may be odd or even, that is, of the form $2n$, or $2n+1$, all prime numbers greater than 3, will be included in $12n \pm 1$, or $12n \pm 5$.

375. *No Algebraical formula can represent prime numbers only.*

Let $p+qx+rx^2+$ &c. be a general algebraical formula; and let it be a prime number when $x=m$; therefore (P) the prime number in that case is

$$p+qm+rm^2+\&c.$$

Now let $x=m+nP$; then

$$p+q(m+nP)+r(m+nP)^2+\&c.$$

is the number; and is equal to

$$p+qm+rm^2+\&c.+MP,$$

(M signifying "some multiple of,") $=P+PM$, which is divisible by P, and therefore not a prime; consequently the formula does not represent prime numbers only.

376. *The number of primes is indefinitely great.*

For if not, let there be a fixed number of them, and let p be the greatest: then

$1.2.3.5.7.11 \ldots\ldots p$ is divisible by each of them,

and $1.2.3.5.7.11 \ldots\ldots p+1$ .... not one of them.

If this latter number, then, be divisible by a prime number, it must be one greater than p; if not, it is itself a prime, (since every number is either a prime, or capable of being resolved into factors which are prime) and is greater than p. Therefore, in either case, there is a prime greater than p; that is, we may not assume any prime to be the greatest; or, the number of primes is indefinitely great.

377. *To determine whether a proposed number be a prime or not.*

It is obvious that this may be done by dividing the proposed number by every number less than itself, beginning with 2, until we have either proved it to be divisible by some one of them without remainder, or that it is a prime, from not being divisible by any one of them. But there is no necessity to proceed so far, as may thus be shewn. If the proposed number (p) be not a prime, then $p = ab$, the product of two other numbers. If then $a > \sqrt{p}$, $b < \sqrt{p}$; and if $a < \sqrt{p}$, $b > \sqrt{p}$. Hence in both cases p is divisible by a number less than the square root of itself. Or it may be that the proposed number is an exact square, in which case it is divisible by its square root. If, therefore, a proposed number be not divisible by some number not greater than the square root of itself, it must be a prime.

378. *To find the number of divisors of a given number.*

Let a, b, c, &c. represent the prime factors of which the given number is composed; and let a be repeated p times; b, q times; c, r times; &c.; so that the number $= a^p b^q c^r$.&c.; then it is evident that it is divisible by each of the quantities,

$$\begin{array}{lll}
1,\ a,\ a^2,\ a^3,\ \dots\dots\ a^p, & p+1 & \text{in number,} \\
1,\ b,\ b^2,\ b^3,\ \dots\dots\ b^q, & q+1 & \dots\ \dots \\
1,\ c,\ c^2,\ c^3,\ \dots\dots\ c^r, & r+1 & \dots\ \dots \\
\quad\&\text{c.} \qquad\qquad \&\text{c.} & \&\text{c.} &
\end{array}$$

and also by the product of any two or more of them, that is, by every term of the continued product of

$$(1 + a + a^2 + \dots + a^p)(1 + b + b^2 + \dots + b^q)(1 + c + c^2 + \dots + c^r).\&\text{c.};$$

$$\begin{aligned}
\text{for } (1 + a + a^2 + \dots a^p)(1 + b + b^2 + \dots + b^q) = {} & 1 + a + a^2 + \dots\dots + a^p \\
& + b + ab + \dots\dots + a^p b \\
& + b^2 + ab^2 + \dots\dots + a^p b^2 \\
& + \dots\dots\dots\dots\dots\dots\dots\dots \\
& + b^q + ab^q + \dots\dots + a^p b^q
\end{aligned}$$

which are all the different divisors (including 1) of $a^p b^q$.

Similarly, if this result be multiplied by $(1 + c + c^2 + \dots + c^r)$, the product will consist of all the different divisors of $a^p b^q c^r$; and so on, if there be more factors of the given number.

Now the *number* of these divisors is obviously $p + 1$ in a^p; in $a^p b^q$ the number is $p + 1$ taken as many times as there are terms in the second series, that is, $(p+1)(q+1)$; in $a^p b^q c^r$ it is $(p+1)(q+1)$ taken $r+1$ times, or $(p+1)(q+1)(r+1)$; and generally

the number of divisors in $a^p b^q c^r$.&c. $= (p+1)(q+1)(r+1)$.&c., including 1 and the number itself.

Ex. Find the number of divisors of 2160.

Here $2160 = 2 \times 1080 = 2^2 \times 540 = 2^3 \times 270 = 2^4 \times 135 = 2^4 \times 3 \times 45 = 2^4 \times 3^2 \times 15 = 2^4 \times 3^3 \times 5$. Therefore the number of divisors

$$= (4+1)(3+1)(1+1) \text{ or } 40.$$

COR. 1. The number of divisors will always be even unless each of the quantities p, q, r, &c. be even, that is, unless the proposed number be a perfect square.

COR. 2. It is also obvious from what has been said above that the ***sum*** of all the different divisors of $a^p b^q c^r$. &c. is the sum of all the terms in the continued product of $(1 + a + a^2 + \ldots + a^p)$ $(1 + b + b^2 + \ldots + b^q)$ $(1 + c + c^2 + \ldots + c^r)$ &c.

$$\text{or} \quad \frac{a^{p+1} - 1}{a - 1} \cdot \frac{b^{q+1} - 1}{b - 1} \cdot \frac{c^{r+1} - 1}{c - 1} \cdot \&\text{c.}$$

379. *If* m *be any prime number, and* N *a number not divisible by* m, *then* $\mathrm{N}^{\mathrm{m}-1} - 1$ *is divisible by* m. (FERMAT'S THEOREM).

For, it is easily seen by the Binomial Theorem, that in the expansion of $(a + b + c + \&\text{c.})^m$ m is a factor of every term except a^m, b^m, c^m, &c.; and that the coefficients are always whole numbers.

But m, being a *prime* number, will not be divisible by any factor in the denominator of a coefficient, and will therefore remain as a factor of each term, when the coefficients are reduced. Therefore we may assume, (m being a prime number),

$$(a + b + c + \&\text{c.})^m = a^m + b^m + c^m + \&\text{c.} + mP.$$

Let, then, $a = b = c = \&\text{c.} = 1$, and the number of them be N; and we have

$$N^m = N + mP,$$

$$\therefore\ N^m - N = mP$$

$$\text{or } N(N^{m-1} - 1) = mP.$$

But, by supposition, N is not divisible by m,

$\therefore$ $N^{m-1} - 1$ is a multiple of m, or is divisible by m.

380. *To prove that, for any positive integral value of* n,

$$n^n - n(n-1)^n + \frac{n(n-1)}{1 \cdot 2}(n-2)^n - \&\text{c.} = 1 \cdot 2 \cdot 3 \ldots n.$$

By the Binomial Theorem,

$$(\epsilon^x - 1)^n = \epsilon^{nx} - n\epsilon^{(n-1)x} + \frac{n(n-1)}{1 \cdot 2}\epsilon^{(n-2)x} - \&\text{c.} \ldots\ldots (1).$$

Also by the Exponential Theorem,

$$(\epsilon^x - 1)^n = \left(1 + x + \frac{x^2}{1 \cdot 2} + \ldots - 1\right)^n = \left(x + \frac{x^2}{1 \cdot 2} + \ldots\right)^n$$

$$= x^n + Ax^{n+1} + Bx^{n+2} + \ldots \text{ by the Binomial Theor.} \ldots (2).$$

Now, the coefficient of x^n in ϵ^{nx} is $\dfrac{n^n}{1 . 2 \ldots n}$,

.... $\epsilon^{(n-1)x}$ ... $\dfrac{(n-1)^n}{1 . 2 \ldots n}$,

.... $\epsilon^{(n-2)x}$ ... $\dfrac{(n-2)^n}{1 . 2 \ldots n}$,

....

and equating coefficients of x^n in (1) and (2),

$$\frac{n^n}{1.2\ldots n} - \frac{n(n-1)^n}{1.2\ldots n} + \frac{n(n-1)}{1\,.\,2}\cdot\frac{(n-2)^n}{1.2\ldots n} - \&c. = 1,$$

$$\text{and } \therefore\ n^n - \frac{n}{1}(n-1)^n + \frac{n(n-1)}{1\,.\,2}(n-2)^n - \&c. = 1\,.\,2\,.\,3\ldots n.$$

Cor. Let p be a prime number, and $p-1=n$, then

$$1\,.\,2\,.\,3\ldots\overline{p-1} = (p-1)^{p-1} - \frac{p-1}{1}(p-2)^{p-1} + \frac{(p-1)(p-2)}{1\,.\,2}(p-3)^{p-1} - \&c.$$

$$= M_1 p + 1 - \frac{p-1}{1}(M_2 p + 1) + \frac{(p-1)(p-2)}{1\,.\,2}(M_3 p + 1) - \&c., \text{by}$$

Fermat's Theorem; $= p\left\{M_1 - (p-1)M_2 + \frac{(p-1)(p-2)}{1\,.\,2}M_3 - \&c.\right\}$

$$+1 - \frac{p-1}{1} + \frac{(p-1)(p-2)}{1\,.\,2} - \&c. \text{ to } \overline{p-1} \text{ terms,}$$

$$= pQ + (1-1)^{p-1} - 1;$$

$\therefore\ 1\,.\,2\,.\,3\ldots\overline{p-1} + 1 = pQ$, or is divisible by p; which is *Wilson's Theorem*, for determining whether any proposed number be a prime or not.

VANISHING FRACTIONS.

381. *To find the value of a fraction when the numerator and denominator are evanescent.*

Since the value of a fraction depends, not upon the absolute, but the relative, magnitude of the numerator and denominator, if in their evanescent state they have a finite ratio, the value of the fraction will be finite. To determine this value, substitute for the variable quantity its magnitude, when the numerator and denominator vanish, increased by another variable quantity; then, after reduction, suppose this latter to decrease without limit, and the value of the proposed fraction will be known.

Ex. 1. Required the value of $\frac{x^2 - a^2}{x - a}$, when $x = a$.

Let $x = a + h$, and the fraction becomes

$$\frac{a^2 + 2ah + h^2 - a^2}{a + h - a} = \frac{2ah + h^2}{h} = 2a + h,$$

and when $h = 0$, or $x = a$, its value is $2a$.

Ex. 2. Required the value of $\dfrac{1-(n+1)\,x^n+n\,x^{n+1}}{(1-x)^2}$ when $x=1$.

Let $x=1+h$, and the fraction becomes

$$\frac{1-(n+1)(1+h)^n+n(1+h)^{n+1}}{h^2}$$

$$=\frac{1-(n+1)\left(1+nh+n\,.\,\frac{n-1}{2}h^2+n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}h^3+\&c.\right)}{h^2}$$

$$+\frac{n\left\{1+(n+1)h+(n+1)\frac{n}{2}h^2+(n+1)\frac{n}{2}\,.\,\frac{n-1}{3}h^3+\&c.\right\}}{h^2}$$

$=n\,.\,\dfrac{n+1}{2}+\dfrac{(n+1)\,n\,(n-1)}{3}h+\&c.$ the remaining terms being multiples of h;

and when $h=0$, or $x=1$, the fraction becomes $n\,.\,\dfrac{n+1}{2}$.

382. It is clear that every fraction, whose terms are made evanescent for a particular value (a) of some quantity (x) contained in them, is capable of being reduced to the form $\dfrac{p\,(x-a)^m}{q\,(x-a)^n}$; and, upon dividing the numerator and denominator by their greatest common factor, the fraction will no longer have *both* its terms evanescent, when $x=a$.

Hence by whatever process this common factor can be discovered, and divided out, the value of the fraction will be found.

A simple algebraical reduction is frequently sufficient for this purpose, as will be shewn in the following Examples.

Ex. 1. Required the value of $\dfrac{1-3x^2+2x^3}{(1-x)^2}$, when $x=1$.

$$\text{Here }\frac{1-3x^2+2x^3}{(1-x)^2}=\frac{1-x^2-2x^2(1-x)}{(1-x)^2}$$

$$=\frac{1+x-2x^2}{1-x}=\frac{1-x^2+x(1-x)}{1-x}$$

$$=1+x+x$$

$$=3,\text{ when } x=1.$$

Ex. 2. Required the value of $\dfrac{x-a+\sqrt{2ax-2a^2}}{\sqrt{x^2-a^2}}$, when $x=a$.

Here $$\frac{x-a+\sqrt{2ax-2a^2}}{\sqrt{x^2-a^2}}=\frac{x-a+\sqrt{2a}\,.\sqrt{x-a}}{\sqrt{x+a}\,.\sqrt{x-a}}$$

$$=\frac{\sqrt{x-a}+\sqrt{2a}}{\sqrt{x+a}}$$

$$=\frac{\sqrt{2a}}{\sqrt{2a}}=1, \text{ when } x=a.$$

Ex. 3. Required the value of $\dfrac{(x+h)^{\frac{m}{n}}-x^{\frac{m}{n}}}{h}$, when $h=0$.

Here $$\frac{(x+h)^{\frac{m}{n}}-x^{\frac{m}{n}}}{h}=\frac{(x+h)^{\frac{m}{n}}-x^{\frac{m}{n}}}{(x+h)-x}$$

$$=(x+h)^{\frac{m}{n}-1}.\frac{1-\left(\dfrac{x}{x+h}\right)^{\frac{m}{n}}}{1-\dfrac{x}{x+h}}$$

$$=(x+h)^{\frac{m}{n}-1}.\frac{1-z^m}{1-z^n}, \text{ if } \left(\frac{x}{x+h}\right)^{\frac{1}{n}}=z,$$

$$=(x+h)^{\frac{m}{n}-1}.\frac{(1-z^m)\div(1-z)}{(1-z^n)\div(1-z)}$$

$$=(x+h)^{\frac{m}{n}-1}.\frac{1+z+z^2+z^3+\ldots z^{m-1}}{1+z+z^2+z^3+\ldots z^{n-1}}$$

$$=x^{\frac{m}{n}-1}.\frac{1+1+1+\ldots \text{to } m \text{ terms}}{1+1+1+\ldots \text{to } n \text{ terms}}, \text{ if } z=1,$$

$$=\frac{m}{n}.x^{\frac{m}{n}-1}, \text{ when } h=0.$$

INFINITE SERIES.

383. Def. An *infinite series* is a series of terms proceeding according to some law, and continued without limit. Thus the series treated of in Art. 290. Cor. 2. is an "infinite series."

Def. The *sum* of an infinite series is the *limit* to which we approach more nearly by adding more terms, (as in Art. 290. Cor. 2.) but cannot be exceeded by adding any number of terms whatever.

Def. A *convergent* series is one which has a *sum* or *limit*, as here defined. A *divergent* series is one which has no such *sum* or *limit*.

Hence every infinite series in Geometrical progression, in which the common ratio is less than 1, is *convergent*.

384. *To determine generally in what cases a series is convergent* *divergent.*

I. Let $a_1 + a_2 + a_3 + a_4 + \dots\dots$ be the series, in which all the terms e positive. Then the sum is equal to

$$a_1 \left\{1 + \frac{a_2}{a_1} + \frac{a_3}{a_1} + \frac{a_4}{a_1} + \dots\dots\right\}$$

$$\text{or} \quad a_1 \left\{1 + \frac{a_2}{a_1} + \frac{a_3}{a_2} \cdot \frac{a_2}{a_1} + \frac{a_4}{a_3} \cdot \frac{a_3}{a_2} \cdot \frac{a_2}{a_1} + \dots\dots\right\}.$$

nd, if each of the quantities $\frac{a_2}{a_1}$, $\frac{a_3}{a_2}$, $\frac{a_4}{a_3}$, &c. be less than some quantity the whole series is less than

$$a_1 \{1 + p + p^2 + p^3 + \dots\dots\}.$$

ut if p be less than 1, the *sum* or *limit* of this latter series is $a_1 \cdot \frac{1}{1-p}$; erefore the proposed series also has a sum or limit *less than* this quantity. ence *an infinite series of positive terms is always convergent, if the tio of each term to the preceding term is less than some assignable antity which is itself less than* 1.

II. Next, let $a_1 - a_2 + a_3 - a_4 + \dots\dots$ be the series, in which the terms e alternately positive and negative, and go on decreasing without limit.

Then, since the series may be written in the two following ways,

$$\overline{a_1 - a_2} + \overline{a_3 - a_4} + \overline{a_5 - a_6} + \dots\dots$$

$$a_1 - \overline{a_2 - a_3} - \overline{a_4 - a_5} - \dots\dots\dots$$

d since $a_3 - a_4$, $a_5 - a_6$, &c.; $a_2 - a_3$, $a_4 - a_5$, &c., are severally positive, is evident that the *sum* or *limit* of the series is greater than $a_1 - a_2$ d less than a_1, that is, the series has a sum or limit; consequently it convergent:—or, *every series, in which the terms continually decrease d are alternately positive and negative, is convergent.*

Cor. If each of the quantities $\frac{b}{a}$, $\frac{c}{b}$, $\frac{d}{c}$, &c. be less than p, the ries $a + bx + cx^2 + dx^3 \dots$ is *convergent* whenever x is less than $\frac{1}{p}$. And p be less than each of the quantities $\frac{b}{a}$, $\frac{c}{b}$, $\frac{d}{c}$, &c. then the series $+ bx + cx^2 + \dots$ is *divergent* for every value of x greater than $\frac{1}{p}$.

Ex. 1. To determine whether $1 + \frac{1}{1} + \frac{1}{1 . 2} + \frac{1}{1 . 2 . 3} + \frac{1}{1 . 2 . 3 . 4} + \dots$ a convergent series.

$$\text{Here} \quad \frac{a_2}{a_1} = 1, \quad \frac{a_3}{a_2} = \frac{1}{2}, \quad \frac{a_4}{a_3} = \frac{1}{3}, \quad \frac{a_5}{a_4} = \frac{1}{4}, \text{ \&c.}$$

each of which ratios, after the second, is less than $\frac{1}{2}$, which is itself less than 1. Therefore the series is *convergent.*

Ex. 2. To determine whether $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots$ be a convergent or divergent series.

$$\text{Here } \frac{a_2}{a_1}=\frac{1}{2},\quad \frac{a_3}{a_2}=\frac{2}{3},\quad \frac{a_4}{a_3}=\frac{3}{4},\ \&\text{c.}$$

and, as these ratios continually increase, and approach towards 1, no quantity can be named which is greater than each of them, and which is itself less than 1. Hence the proposed series is *divergent.*

385. Another method of determining the convergency or divergency of series is to find the limit of the sum of the series after the first n terms; which also determines the limits of the error arising from taking any number of terms instead of the whole series. Thus,

Ex. 1. In the series $1+\frac{1}{1}+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots\ldots$ the sum of the series after n terms

$$=\frac{1}{\underline{|n}}+\frac{1}{\underline{|n+1}}+\frac{1}{\underline{|n+2}}+\ldots\ldots\ldots\ldots\ldots\ldots$$

$$=\frac{1}{\underline{|n}}\cdot\left\{1+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\ldots\ldots\right\}$$

$$<\frac{1}{\underline{|n}}\cdot\left\{1+\frac{1}{n}+\frac{1}{n^2}+\frac{1}{n^3}+\ldots\ldots\right\}$$

$$<\frac{1}{\underline{|n}}\cdot\frac{1}{1-\frac{1}{n}}$$

$$<\frac{1}{\underline{|n-1}}\cdot\frac{1}{n-1},\quad\text{or }<\frac{1}{1.2.3\ldots\overline{n-1}}\cdot\frac{1}{n-1}.$$

But this quantity decreases as n increases; and, by increasing n without limit, it may be made less than any assignable quantity. Therefore the series is *convergent.* And if n terms be taken for the whole series, the error is less than

$$\frac{1}{1.2.3\ldots\overline{n-1}}\cdot\frac{1}{n-1}.$$

Ex. 2. In the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots$ the sum of the series after n terms

$$=(-1)^n\cdot\left\{\frac{1}{n+1}-\frac{1}{n+2}+\left(\frac{1}{n+3}-\frac{1}{n+4}\right)+\ldots\ldots\right\},$$

$$\text{or }=(-1)^n\cdot\left\{\frac{1}{n+1}-\left(\frac{1}{n+2}-\frac{1}{n+3}\right)-\left(\frac{1}{n+4}-\frac{1}{n+5}\right)-\ldots\right\},$$

and, since the quantities within the smaller brackets are all positive, this sum

$$>(-1)^n.\left\{\frac{1}{n+1}-\frac{1}{n+2}\right\},$$

$$\text{and} \quad <(-1)^n.\frac{1}{n+1},$$

both which quantities are diminished without limit as n is increased.

Hence the series is *convergent;* and if n terms be taken for the whole series, the error is less than the $\overline{n+1}^{\text{th}}$ term.

Ex. 3. In the series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\ldots$ the sum of the series after n terms

$$=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots\ldots$$

$$=\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\ldots\ldots\ldots+\frac{1}{2n}$$

$$+\frac{1}{2n+1}+\frac{1}{2n+2}+\frac{1}{2n+3}+\ldots+\frac{1}{4n}$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$>\frac{1}{2n}+\frac{1}{2n}+\text{\&c. to } n \text{ terms}+\frac{1}{4n}+\frac{1}{4n}+\text{\&c. to } 2n \text{ terms}+\ldots\ldots$$

$$>n.\frac{1}{2n}+2n.\frac{1}{4n}+4n.\frac{1}{8n}+\ldots\ldots\ldots\ldots$$

$$>\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$>$ any assignable quantity.

Hence the series is *divergent.*

386. *In the series* $a_1x+a_2x^2+a_3x^3+\ldots\ldots$ in inf. *such a value may be given to* x, *that the value of the whole series shall be less than any proposed quantity* p.

Let k be the greatest of the coefficients $a_1, a_2, a_3, \ldots\ldots$, then the whole series is less than

$$kx+kx^2+kx^3+\ldots\ldots \textit{ in inf.}$$

$$<k.\frac{x}{1-x}, \text{ if } x<1;$$

hence that which is required is done, if x be such a value that

$$\frac{kx}{1-x}<p,$$

that is, $kx < p - px$,

$$\text{or} \quad x < \frac{p}{p+k}.$$

Cor. Hence also in the series $a_0 + a_1 x + a_2 x^2 + \ldots\ldots$ *in inf.* such a value may always be given to x, that the first term is greater than the sum of all the other terms. This value of x will be any quantity less than $\frac{a_0}{a_0 + k}$.

RECURRING SERIES.

387. If each succeeding term of a decreasing Infinite Series bear an invariable relation to a certain number of the preceding terms, the series is called a Recurring Series, and its sum may be found.

Let $a + bx + cx^2 +$ &c. be the proposed series; call its terms A, B, C, D, &c. and let

$$C = fxB + gx^2A, \quad D = fxC + gx^2B, \text{ \&c.}$$

where $f + g$ is called the *scale of relation*; then, by the supposition,

$$\begin{aligned} A &= A \\ B &= B \\ C &= fxB + gx^2A \\ D &= fxC + gx^2B \\ E &= fxD + gx^2C \\ \text{\&c.} &= \quad \text{\&c.} \end{aligned}$$

and, if the whole sum $A + B + C + D +$ &c. *in inf.* $= S$, we have

$$S = A + B + fx.(S - A) + gx^2.S,$$

$$\text{or } S - fxS - gx^2S = A + B - fxA;$$

$$\therefore S = \frac{A + B - fxA}{1 - fx - gx^2}.$$

In the same manner, if the scale of relation be $f + g +$ &c. to n terms, the sum of the series is

$$\frac{A+B+C\ldots\text{to } n \text{ terms} - fx\{A+B\ldots\text{to } \overline{n-1} \text{ terms}\} - gx^2\{A+\ldots\text{to } \overline{n-2} \text{ terms}\} - \text{\&c.}}{1 - fx - gx^2 - hx^3 \ldots \text{ to } (n+1) \text{ terms.}}$$

Ex. 1. To find the sum of the infinite series $1 + 3x + 9x^2 + \&c.$ hen x is less than $\frac{1}{3}$.

$$\text{Here } f = 3, \text{ and the sum} = \frac{1}{1 - 3x}.$$

Ex. 2. To find the sum of the infinite series $1 + 2x + 3x^2$ $4x^3 + \&c.$ when x is less than 1.

Here $f = 2$, $g = -1$, and

$$\text{the sum} = \frac{1 + 2x - 2x}{1 - 2x + x^2} = \frac{1}{(1 - x)^2}.$$

If $x =$ or > 1, the series is infinite; yet we know that it arises rom the division of 1 by $(1 - x)^2$, and the sum of n terms may be etermined.

The series after the first n terms becomes

$$(n + 1)\,x^n + (n + 2)\,x^{n+1} + (n + 3)\,x^{n+2} + \&c.$$

n which the scale of relation, as before, is $2 - 1$; and therefore he series arises from the fraction

$$\frac{(n + 1)\,x^n + (n + 2)\,x^{n+1} - 2\,(n + 1)\,x^{n+1}}{1 - 2x + x^2},$$

$$\text{or } \frac{(n + 1)\,x^n - nx^{n+1}}{(1 - x)^2};$$

$$\therefore\ 1 + 2x + 3x^2 + \&c. \text{ to } n \text{ terms} = \frac{1 - (n + 1)\,x^n + nx^{n+1}}{(1 - x)^2}.$$

Cor. If the sign of x be changed, $1 - 2x + 3x^2 - \&c.$ to n terms

$$= \frac{1 \mp (n + 1)\,x^n \mp nx^{n+1}}{(1 + x)^2},$$

here the upper or lower sign is to be used, according as n is an ven or odd number.

Ex. 3. To find the sum of n terms of the series $1 + 3x + 5x^2 + 7x^3 + \&c.$

Suppose $f + g$ to be the scale of relation;

then $3f + g = 5$, and $5f + 3g = 7$; hence $f = 2$, and $g = -1$;

and, by trial, it appears that the scale of relation is properly determined;

$$\text{hence } S = \frac{1 + 3x - 2x}{1 - 2x + x^2} = \frac{1 + x}{(1 - x)^2}.$$

After n terms, the series becomes

$$(2n + 1)\,x^n + (2n + 3)\,x^{n+1} + (2n + 5)\,x^{n+2} + \&c.$$

which arises from the fraction

$$\frac{(2n + 1)\,x^n + (2n + 3)\,x^{n+1} - 2\,(2n + 1)\,x^{n+1}}{1 - 2x + x^2};$$

$$\text{or } \frac{(2n + 1)\,x^n - (2n - 1)\,x^{n+1}}{(1 - x)^2};$$

hence $1 + 3x + 5x^2 + 7x^3 + \&c.$ to n terms,

$$= \frac{1 + x - (2n + 1)\,x^n + (2n - 1)\,x^{n+1}}{(1 - x)^2}.$$

Ex. 4. To find the sum of $1 + 2x + 3x^2 + 5x^3 + 8x^4 + \&c.$ *in inf.* when the series converges.

In this case the scale of relation is $1 + 1$, and consequently the sum is

$$\frac{1 + 2x - x}{1 - x - x^2} = \frac{1 + x}{1 - x - x^2}.$$

If x becomes negative,

$$1 - 2x + 3x^2 - 5x^3 + \&c. \textit{ in inf.} = \frac{1 - x}{1 + x - x^2}.$$

Ex. 5. To find the sum of n terms of the series

$$(n - 1)\,x + (n - 2)\,x^2 + (n - 3)\,x^3 + \&c.$$

The scale of relation is $2 - 1$; therefore the sum *in inf.* is

$$\frac{(n - 1)\,x + (n - 2)\,x^2 - 2\,(n - 1)\,x^2}{(1 - x)^2}, \text{ or } \frac{(n - 1)\,x - nx^2}{(1 - x)^2}.$$

After n terms, the series becomes $- x^{n+1} - 2x^{n+2} - \&c.$ the sum of which is found in the same manner to be $\dfrac{- x^{n+1}}{(1 - x)^2}$;

$\therefore$ $(n-1)x+(n-2)x^2+(n-3)x^3+$ &c. to n terms,

$$=\frac{(n-1)x-nx^2+x^{n+1}}{(1-x)^2}.$$

COR. Hence $\frac{(n-1)x}{n}+\frac{(n-2)x^2}{n}+$ &c. to n terms,

$$=\frac{(n-1)x-nx^2+x^{n+1}}{n(1-x)^2}.$$

Ex. 6. To find the sum of n terms of the series

$$1^2+2^2x+3^2x^2+4^2x^3+\text{\&c.}$$

Let the scale of relation be $f+g+h$; then

$$9f+4g+h=16,$$
$$16f+9g+4h=25,$$
$$25f+16g+9h=36.$$

From these equations we obtain $f=3$, $g=-3$, $h=1$, which alues, when substituted, produce the successive terms of the proosed series; therefore

$$=\frac{1+4x+9x^2-3x-12x^2+3x^2}{1-3x+3x^2-x^3}=\frac{1+x}{(1-x)^3},$$ the sum of the eries *in inf.* when x is less than 1.

After the first n terms the series becomes

$$(n+1)^2x^n+(n+2)^2x^{n+1}+(n+3)^2x^{n+2}+\text{\&c.}$$

f which the sum is

$$\frac{(n+1)^2x^n-(2n^2+2n-1)x^{n+1}+n^2x^{n+2}}{(1-x)^3};$$

nd consequently the sum of n terms of the series is

$$\frac{1+x-(n+1)^2x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}.$$

On this subject the reader may consult De Moivre's *Misc. Analyt.* p. 72; and Euler's *Analys. Infinit.* C. XIII.

LOGARITHMS.

388. Def. If there be a series of magnitudes

$$a^0,\ a^1,\ a^2,\ a^3, \dots a^x;\ a^{-1},\ a^{-2},\ a^{-3}, \dots a^{-y},$$

the indices, $0,\ 1,\ 2,\ 3, \dots x;\ \ -1,\ -2,\ -3, \dots -y,$

are called the measures of the ratios of those magnitudes to 1, or the *Logarithms* of the magnitudes, for the reason assigned in Art. 230. Thus x, the *Logarithm* of any number n, is such a quantity, that $a^x = n$.

Here a may be assumed at pleasure, and is called the *base;* and for every different value so assumed a different *system* of logarithms will be formed. In the common Tabular logarithms a is 10, and consequently $0, 1, 2, 3, \dots x$, are the logarithms of 1, 10, 100, 1000, ... 10^x in that system.

389. Cor. 1. Since the tabular logarithm of 10 is 1, the logarithm of a number between 1 and 10 is less than 1; and, in the same manner, the logarithm of a number between 10 and 100 is between 1 and 2; of a number between 100 and 1000 is between 2 and 3; &c.

These logarithms are also real quantities, to which approximation, sufficiently accurate for all practical purposes, may be made.

Thus, if x be the logarithm of 5, then $10^x = 5$; let $\frac{2}{3}$ be substituted for x, and $10^{\frac{2}{3}}$ is found to be less than 5, therefore $\frac{2}{3}$ is less than the logarithm of 5; but $10^{\frac{3}{4}}$ is greater than 5, or $\frac{3}{4}$ is greater than the logarithm of 5; thus it appears that there is a value of x between $\frac{2}{3}$ and $\frac{3}{4}$, such that $10^x = 5$; the value set down in the Tables* is 0·69897, and $10^{0.69897} = 5$, nearly.

390. Cor. 2. Since $a^0 = 1$, $b^0 = 1$, &c. in *any* system the logarithm of 1 is 0. Also since $a^1 = a$, the logarithm of the base is always 1.

The method of finding the logarithms of the natural numbers, or forming a Table*, is explained in Treatises on Trigonometry. See Snowball's Trigonometry, Appendix I.

Def. If n be any number, $\log_a n$ signifies the logarithm of n to base a; and $\log n$ the logarithm of n to *any* base.

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* Tables of Logarithms have been lately published in a very cheap and convenient form by Taylor and Walton, London, under the superintendence of the Society for the Diffusion of Useful Knowledge.

391. *In the same system the sum of the logarithms of two numbers is the logarithm of their product; and the difference of the logarithms is the logarithm of their quotient.*

Let $x = \log_a n$, and $y = \log_a n'$; then $a^x = n$, and $a^y = n'$; hence $a^{x+y} = nn'$, and $a^{x-y} = \frac{n}{n'}$; or $x + y$ is $\log_a \overline{nn'}$, and $x - y$ is $\log_a \frac{n}{n'}$; that is, $\log_a \overline{nn'} = \log_a n + \log_a n'$; and $\log_a \frac{n}{n'} = \log_a n - \log_a n'$.

Ex. 1. $\text{Log}\ \overline{3 \times 7} = \log 3 + \log 7$.

Ex. 2. $\text{Log}\ \overline{pqr} = \log \overline{pq} + \log r = \log p + \log q + \log r$.

Ex. 3. $\text{Log}\ \frac{5}{7} = \log 5 - \log 7$.

Ex. 4. $\text{Log}_{10} 0{\cdot}6 = \log_{10} \frac{6}{10} = \log_{10} 6 - \log_{10} 10 = 0{\cdot}77815 - 1$.

Ex. 5. $\text{Log}_{10} 0{\cdot}006 = \log_{10} 6 - \log_{10} 10^3 = 0{\cdot}77815 - 3$.

The last two results are usually written $\bar{1}{\cdot}77815$, $\bar{3}{\cdot}77815$.

392. *If the logarithm of a number be multiplied by* n, *the product is the logarithm of that number raised to the* nth *power.*

Let N be the number whose logarithm is x, or $a^x = N$; then $a^{nx} = N^n$; that is, nx is the log. of N^n, or $\log_a N^n = n \,.\, \log_a N$.

Exs. $\text{Log}\ (13)^5 = 5 \times \log 13$. $\text{Log}\ b^x = x \times \log b$.

Cor. $\text{Log}\ (a^m b^n c^p \dots) = m \log a + n \log b + p \log c + \dots$

Ex. $\text{Log}\ \sqrt{a^2 - x^2} = \log (\sqrt{a + x} \,.\, \sqrt{a - x}) = \frac{1}{2} \log \overline{a + x} + \frac{1}{2} \log \overline{a - x}$.

393. *If the logarithm of a number be divided by* n, *the quotient is the logarithm of the* nth *root of that number.*

Let $a^x = N$, then $a^{\frac{x}{n}} = N^{\frac{1}{n}}$, or $\frac{x}{n}$ is the log. of $N^{\frac{1}{n}}$, that is, $\log_a N^{\frac{1}{n}} = \frac{1}{n} \,.\, \log_a N$.

Exs. $\text{Log}\ 5^{\frac{1}{4}} = \frac{1}{4} \times \log 5$. $\text{Log}\ \sqrt{\frac{a}{b}} = \frac{1}{2} \log a - \frac{1}{2} \log b$.

394. The utility of a Table of logarithms in arithmetical calculations will from hence be manifest; the multiplication and division of numbers being performed by the addition and subtraction

of these artificial representatives; and the involution or evolution of numbers by multiplying or dividing their logarithms by the indices of the powers or roots required.

Also the value of x which satisfies an equation of the form $a^x = b$, may be found since $x \,.\, \log a = \log b$, and $\therefore\ x = \dfrac{\log b}{\log a}$.

But much practice will be needed before the student will be able to make a satisfactory use of the Tables, and he will be required to take the subject in hand with great earnestness and determination. He is referred to the Treatise before mentioned, *Snowball's Trigonometry*, where he will find every requisite direction. A few easy examples are subjoined.

Ex. 1. Required the value of $\sqrt[7]{3047}$.

$$\text{Here}\quad \log_{10}\sqrt[7]{3047} = \frac{1}{7}\log_{10} 3047 = \frac{3{\cdot}48387}{7}$$

$$= 0{\cdot}49769 = \log_{10} 3{\cdot}1456;$$

$\therefore$ 3·1456 is the root required.

Ex. 2. Let the value of $\sqrt[5]{7\sqrt{2}\times\sqrt[3]{3}}$ be required.

The log. of the proposed quantity to base 10 is

$$\tfrac{1}{5}\left\{\log_{10} 7 + \tfrac{1}{2}\log_{10} 2 + \tfrac{1}{3}\log_{10} 3\right\} \text{ (Arts. 391, 393.)}$$

$$\begin{aligned}
\text{And by the Tables}\quad \log_{10} 7 &= 0{\cdot}845098\\
\tfrac{1}{2}\log_{10} 2 &= 0{\cdot}150515\\
\tfrac{1}{3}\log_{10} 3 &= 0{\cdot}1590404\\
\hline
5\,)\ &1{\cdot}1546534\\
\hline
0{\cdot}2309306 &= \log_{10} 1{\cdot}70188 \text{ \&c.}
\end{aligned}$$

$\therefore$ the value required is 1·07188 &c.

Ex. 3. Find a fourth proportional to the 6th power of 9, the 4th power of 7, and the 5th power of 5.

Let x be the required number; then

$$9^6 : 7^4 :: 5^5 : x, \text{ and } x = \frac{7^4 \times 5^5}{9^6}.$$

$$\begin{aligned}
\therefore\ \log x &= 4\log 7 + 5\log 5 - 6\log 9\\
&= 3{\cdot}38040 + 3{\cdot}49485 - 5{\cdot}72544\\
&= 1{\cdot}14981 = \log 14{\cdot}12 \text{ nearly;}
\end{aligned}$$

$$\therefore\ x = 14{\cdot}12 \text{ nearly.}$$

Ex. 4. Given $s = a \,.\, \dfrac{r^n - 1}{r - 1}$; required the value of n.

$$\text{Here}\quad ar^n = s(r-1)+a$$

$$\therefore\ \log a + \log r^n = \log \overline{s(r-1)+a}$$

$$n\,.\log r = \log \overline{s(r-1)+a} - \log a\,;$$

$$\therefore\ n = \frac{\log (s\,.\,\overline{r-1}+a) - \log a}{\log r}\,;$$

om which we may obtain the number of terms in any Geometric Pro- :ession, when the first term, common ratio, and sum, are given.

Ex. 5. Given $2^x = 1976$; find the value of x.

Here $x\,.\log 2 = \log 1976$,

$$\therefore\ x = \frac{\log 1976}{\log 2} = \frac{3{\cdot}29579}{0{\cdot}30103} = 10{\cdot}9 \text{ nearly.}$$

Ex. 6. Given $a^{b^x} = c$; find x*.

$$\text{Let}\quad b^x = y,\ \text{then}\ x\,.\log b = \log y,\ \text{and}\ x = \frac{\log y}{\log b}.$$

$$\text{Also}\quad a^y = c,\ \therefore\ y\,.\log a = \log c,\ \text{and}\ y = \frac{\log c}{\log a}\,;$$

$$\therefore\ x = \frac{\log(\log c) - \log(\log a)}{\log b}.$$

INTEREST AND ANNUITIES.

395. **Def.** *Interest* is the consideration paid for the use of ioney which belongs to another. The *rate* of interest is the con- deration paid for the use of a certain sum for a certain time, as of 1 for one year.

When the interest of the *Principal* alone, or sum lent, is taken, is called *Simple Interest*; but if the interest, as soon as it be- ›mes due, be added to the principal, and interest be charged upon ie whole, it is called *Compound Interest.*

The *Amount* is the whole sum due at the end of any time, In- rest and Principal together.

Discount is the abatement made for the payment of money :fore it becomes due.

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* In this Ex. by a^{b^x} is meant a raised to the power expressed by b^x, and not a^b raised the x^{th} power.

SIMPLE INTEREST.

396. *To find the Amount of a given sum, in any time, at simple interest.*

Let P be the principal, in pounds,

n the No. of years for which the interest is to be calculated[*].

r the interest of £1 for one year[†],

M the amount.

Then, since the interest of a given sum, at a given rate, must be proportional to the time, 1 (year) : n (years) :: $r : nr$, the interest of £1 for n years; and the interest of P£ must be P times as great, or Pnr; therefore the amount $M = P + Pnr$.

397. From this simple equation, any three of the quantities P, n, r, M being given, the fourth may be found; thus

$$P = \frac{M}{1 + nr}; \quad n = \frac{M - P}{Pr}; \quad r = \frac{M - P}{Pn}.$$

Ex. What sum must be paid down to receive 600£, at the end of nine months, allowing 5 per cent. abatement? Or, which is the same thing, what principal P will in nine months amount to 600£, allowing interest at the rate of 5 per cent. per annum?

In this case $M = 600$, $n = \frac{3}{4} = 0{\cdot}75$, $r = \frac{5}{100} = 0{\cdot}05$; hence

$$P = \frac{M}{1 + nr} = \frac{600}{1 + 0{\cdot}75 \times 0{\cdot}05} = £578{\cdot}313 = £578.\ 6s.\ 3d.$$

COMPOUND INTEREST.

398. *To find the amount of a given sum in any time at compound interest.*

Let $R = 1$£ together with its interest for a year; then at the end of the first year, R becomes the principal, or sum due;

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[*] When days, weeks, or months, not making an exact number of years, enter the calculation, n is fractional.

[†] It must always be borne in mind that r is not the rate *per cent.* but only the hundredth part of it. Thus for 4 per cent. $r = 0{\cdot}04$£, for 5 per cent. $r = 0{\cdot}05$£; and so on.

The amount at the end of the 2d year = amount of R£ in 1 year

$$= R \times R = R^2,$$

The amount at the end of the 3d year = amount of R^2 in 1 year

$$= R^2 \times R = R^3;$$

d so on; so that R^n is the amount of 1£ in n years. And if P£ e the principal, the amount must be P times as great, or

$$M = PR^n.$$

Cor. 1. From this equation we have

$$P = \frac{M}{R^n}, \quad n = \frac{\log M - \log P}{\log R}, \quad \text{and } R = \left(\frac{M}{P}\right)^{\frac{1}{n}}.$$

Cor. 2. The interest $= M - P = PR^n - P = P\{R^n - 1\}$.

Ex. 1. What must be paid down to receive 600£ at the end f 3 years, allowing 5 per cent. per annum compound interest?

In this case $R = 1{\cdot}05$, $n = 3$, $M = 600$; therefore

$$P = \frac{M}{R^n} = \frac{600}{(1{\cdot}05)^3} = £518{\cdot}302 = £518.\ 6s.\ 0\tfrac{1}{2}d.$$

Ex. 2. Find the amount of 5£ in $2\frac{1}{2}$ years at 3 per cent., compound terest.

Here $P = 5$, $R = 1{\cdot}03$, $n = \frac{5}{2}$,

$$\begin{aligned}\therefore\ R^n &= (1{\cdot}03)^{\frac{5}{2}} = (1 + 0{\cdot}03)^{\frac{5}{2}}\\ &= 1 + \frac{5}{2} \times 0{\cdot}03 + \frac{5}{2}.\frac{\frac{5}{2} - 1}{2} \times (0{\cdot}03)^2 + \dots\dots\\ &= 1 + 0{\cdot}075 + 0{\cdot}0017 + \dots\dots\\ &= 1{\cdot}0767 \text{ nearly};\end{aligned}$$

$\therefore\ PR^n = 5 \times 1{\cdot}0767 = 5{\cdot}3835 = 5£.\ 7s.\ 8d.$, the amount required.

399. When compound interest is named, it is usually meant that terest is payable only at the end of each year; but there may be cases which the interest is due half-yearly, quarterly, &c.; and then the nount found in the last Article will be altered. Thus, if r be the interest 1£ paid at the end of a year, it has been shewn that the amount of P£ the end of n years $= P(1 + r)^n$.

But if $\frac{r}{2}$ be the interest of 1£ paid at the end of each half-year, then

$$1+\frac{r}{2} = \text{the amount of } 1\pounds \text{ for half a year,}$$

$$\left(1+\frac{r}{2}\right)^2 = \ldots\ldots\ldots\ldots\ldots\ldots \quad 1 \text{ year,}$$

$$\left(1+\frac{r}{2}\right)^3 = \ldots\ldots\ldots\ldots\ldots\ldots \quad 1\tfrac{1}{2} \text{ years,}$$

$$\left(1+\frac{r}{2}\right)^4 = \ldots\ldots\ldots\ldots\ldots\ldots \quad 2 \text{ years;}$$

$$\ldots\ldots = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\left(1+\frac{r}{2}\right)^{2n} = \ldots\ldots\ldots\ldots\ldots\ldots \quad n \text{ years;}$$

$$\therefore \text{ amount of } P\pounds = P\left(1+\frac{r}{2}\right)^{2n}.$$

Similarly, if $\frac{r}{4}$ be the interest of 1£ paid at the end of each quarter,

$$\text{amount of } P\pounds \text{ in } n \text{ years} = P\left(1+\frac{r}{4}\right)^{4n}.$$

And, generally, if interest be considered due q times a year, at equal intervals, each payment for 1£ being $\frac{r}{q}$,

$$\text{amount of } P\pounds \text{ in } n \text{ years} = P\left(1+\frac{r}{q}\right)^{nq}.$$

Ex. Find the amount of 100£ in 1 year at 5 per cent. *per annum*, when the interest is due, and converted into principal, at the end of each *half-year*.

$$\text{Here} \quad P = 100, \quad r = 0{\cdot}05, \quad n = 1,$$

$$\therefore \text{ Amount required} = 100 \times (1+0{\cdot}025)^2 = 105{\cdot}0625\pounds.$$

$$= 105\pounds.\ 1s.\ 3d.$$

400. *Required the amount of a given sum at compound interest, the interest being supposed due every instant.*

The interest being paid q times per annum, by last Art., the amount

$$M = P\left(1+\frac{r}{q}\right)^{nq}$$

$$= P\left\{1 + nq\cdot\frac{r}{q} + \frac{nq(nq-1)}{1\ .\ 2}\cdot\frac{r^2}{q^2} + \ldots\right\}$$

$$= P\left\{1 + nr + \frac{n\left(n-\frac{1}{q}\right)}{1\ .\ 2}r^2 + \ldots\right\}.$$

Let q be indefinitely great, that is, the intervals between the payments indefinitely small, then, neglecting $\frac{1}{q}$ and its powers,

$$M = P\{1 + nr + \frac{n^2r^2}{1.2} + \frac{n^3r^3}{1.2.3} + \dots\}$$

$$= P\epsilon^{nr},\ \epsilon \text{ being } 2{\cdot}7182818. \quad (\text{Art. } 325.)$$

401. *To determine the advantage, when compound interest is reckoned, of having interest paid half-yearly, quarterly, &c. instead of yearly.*

It appears from Art. 399 that the advantage per 1 £ *for a year*, when interest is paid half-yearly, and the half-yearly payment is half the yearly one,

$$= \left(1 + \frac{r}{2}\right)^2 - (1 + r) = 1 + r + \frac{r^2}{4} - (1 + r)$$

$$= \frac{r^2}{4}.$$

In the case of quarterly payments of interest, the advantage

$$= \left(1 + \frac{r}{4}\right)^4 - (1 + r)$$

$$= 1 + r + \frac{3r^2}{8} + \frac{r^3}{16} + \frac{r^4}{256} - (1 + r)$$

$$= \frac{3r^2}{8} \text{ nearly, } \because r \text{ is a small fraction.}$$

And generally, when the interest is paid q times a year, the advantage

$$= \left(1 + \frac{r}{q}\right)^q - (1 + r)$$

$$= 1 + r + \frac{q(q-1)}{1\ .\ 2} . \frac{r^2}{q^2} + \&\text{c.} - (1 + r)$$

$$= \frac{q-1}{2q} . r^2 \text{ nearly.}$$

Hence it appears, that for a single year the advantage of having interest paid frequently is very small. But it increases as the number of years increases, and is expressed in n years, for every 1 £, (when interest is paid q times a year at equal intervals, $\frac{r}{q}$ being the payment per 1 £,) by

$$\left(1 + \frac{r}{q}\right)^{nq} - (1 + r)^n.$$

402. It must be observed always, when interest is paid q times per annum, each payment being $\frac{r}{q}$ for every 1 £, that the true *annual* rate of interest is not r, but $\left(1 + \frac{r}{q}\right)^q - 1$, since this expresses the value of the interest for 1 £ for a year.

Prob. In what time will any sum of money double itself, at any given rate of interest simple or compound?

I. In the case of simple interest $M = P + Pnr$,

$$\therefore \text{ here } 2P = P + Pnr,$$

$$\text{or } n = \frac{1}{r}, \text{ the number of years required.}$$

II. In the case of compound interest paid yearly $M = P(1+r)^n$,

$$\therefore \text{ here } 2P = P(1+r)^n,$$

$$\text{or } (1+r)^n = 2;$$

$$\therefore n \quad \frac{\log 2}{\log 1 + r}, \text{ the number of years required.}$$

The following Table, calculated from these two results, and shewing the several times in which any sum will double itself at the rates of interest there given, is taken from Baily's *Doctrine of Interest and Annuities*, and will furnish good practice to the learner, who will verify it by means of the Logarithmic Tables.

Rate of Interest.	No. of Years at	
	Simple nterest.	Compound Interest.
2	50·	35·00278878
$2\frac{1}{2}$	40·	28·07103453
3	33·33333333	23·44977225
$3\frac{1}{2}$	28·57142857	20·14879168
4	25·	17·67298769
$4\frac{1}{2}$	22·22222222	15·74730184
5	20·	14·20669908
6	16·66666667	11·89566105
7	14·28571429	10·24476835
8	12·5	9·00646834
9	11·11111111	8·04323173
10	10·	7·27254090

DISCOUNT.

403. Discount is said in Art. 395 to be "the abatement made for the payment of money before it becomes due", and although in ordinary business the *quantity* of such abatement is generally according to private contract, there is besides a true mathematical discount which affords exact justice both to the payer and to the receiver.

It is clear that if A receives from B a sum of money n years before is due, n being whole or fractional, A is benefited by the interest of at money for the time; and therefore, in justice, B ought to receive abatement such, that, if put out to interest until the proper time of yment arrives, it will *amount* to the interest of the debt for that time.

Hence, if D be the discount payable on P£ due at the expiration n years, reckoning simple interest,

$$D + \text{interest of } D \text{ for time } n = \text{interest of } P \text{ for the same time},$$

$$\text{that is,}\quad D + Dnr = Pnr,$$

$$\therefore\ D = \frac{Pnr}{1+nr}.$$

Cor. 1. If D be the discount, and I the interest, of P£ for any time,

$$D = \frac{I}{1+\frac{I}{P}} = \frac{PI}{P+I},$$

$$\text{or}\quad \frac{1}{D} = \frac{1}{P} + \frac{1}{I}.$$

Cor. 2. If n be sufficiently great that compound interest may be ckoned, then

$$D(1+r)^n = P(1+r)^n - P;$$

$$\therefore\ D = P - \frac{P}{(1+r)^n}.$$

Cor. 3. Compound interest being reckoned, if D be the discount, d I the interest, on P£,

$$D(1+r)^n = I = P(1+r)^n - P;$$

$$\therefore\ I = \frac{PI}{D} - P,$$

$$\text{or}\quad \frac{1}{D} = \frac{1}{P} + \frac{1}{I},\ \text{as for simple interest.}$$

Cor. 4. The *Present Value* of P£, due n years hence $= P - D$; d therefore, reckoning simple interest $= P - \dfrac{Pnr}{1+nr} = \dfrac{P}{1+nr}$;

or reckoning compound interest $= \dfrac{P}{(1+r)^n}$.

EQUATION OF PAYMENTS.

404. When various sums of money due at different times are to paid, it may be required to know the time at which they may all be id together, without injury to either debtor or creditor. To determine s time, which is called the *Equated Time*, it is clear that we must

suppose the interest of the sums paid after they are due to be together equal to the discount of the sums paid before they are due, the debtor being entitled to discount for that which is paid before, and the creditor to interest for that which is paid after, it becomes due.

405. *To find the equated time of payment of two sums due at different times, reckoning simple interest.*

Let P, p, be two sums due at the end of times T, t, respectively; r the rate of interest, and x the equated time; then supposing $T > t$, the interest of p for the time $x - t$ must be equal to the discount of P for the time $T - x$, or

$$p(x-t)r = \frac{P(T-x)r}{1+(T-x)r}, \text{ (Arts. 396, 403.)}$$

whence we may obtain the quadratic equation

$$x^2 - \frac{pr(T+t) + P + p}{pr} \cdot x + \frac{prTt + PT + pt}{pr} = 0,$$

from which x may be found by the usual method.

406. Since the above method does not furnish any simple Rule, and is more complicated as the number of payments is increased, another method is generally used, although incorrect, which is founded on the supposition that the interest of the sums paid after they are due should be equal to the *interest* (not the *discount*) of those which are paid before they are due.—Thus, if P and p be the sums due at the end of times T and t, and x the equated time required,

$$p(x-t)r = P(T-x)r;$$

$$\therefore\ x = \frac{PT + pt}{P+p}.$$

Or, more generally, let P_1, P_2, P_3, &c. be the sums due after the equated time, at the end of times T_1, T_2, T_3, &c. and p_1, p_2, p_3, &c. the sums due before, at the end of times t_1, t_2, t_3, &c. then we have

$$p_1(x-t_1)r + p_2(x-t_2)r + p_3(x-t_3)r + \text{\&c.}$$
$$= P_1(T_1-x)r + P_2(T_2-x)r + P_3(T_3-x)r + \text{\&c.}$$

$$\text{and } \therefore\ x = \frac{P_1T_1 + P_2T_2 + P_3T_3 + \text{\&c.} + p_1t_1 + p_2t_2 + p_3t_3 + \text{\&c.}}{P_1 + P_2 + P_3 + \text{\&c.} + p_1 + p_2 + p_3 + \text{\&c.}},$$

which furnishes a simple rule easy of application.

By this rule a small advantage is given to the payer, because he reckons on his side the interest, instead of the discount, of those sums which he pays before they are due, whilst the opposite side of the account is confined to strict accuracy. That the interest of any sum is greater than the discount, if not obvious, may be seen from Art. 403, Cor. 1.

407. Another method of finding the Equated time is to find the *Present Value* of each payment, and make the sum of them equal to

the Present Value of the sum of the several payments supposed due at the Equated time. Thus, if P, p, be due at the end of times T, t, respectively, r the rate of interest, and x be the equated time,

$$\text{Present value of } P = \frac{P}{1+Tr}, \text{ (Art. 403, Cor. 4)}$$

$$\text{..............} \text{ of } p = \frac{p}{1+tr}$$

$$\text{..........} \text{ of } P+p = \frac{P+p}{1+xr},$$

$$\text{and } \frac{P}{1+Tr} + \frac{p}{1+tr} = \frac{P+p}{1+xr},$$

from which simple equation with respect to x, we get

$$x = \frac{PT + pt + r(P+p)Tt}{P+p+r(Pt+pT)}.$$

Cor. If the quantities multiplied by r be neglected, since r is generally a very small fraction, we have

$$x = \frac{PT+pt}{P+p}, \text{ which is the common rule.}$$

ANNUITIES.

408. *To find the Amount of an annuity or pension left unpaid any number of years, allowing simple interest upon each sum or pension from the time it becomes due.*

Let A be the annuity; then at the end of the first year A becomes due, and at the end of the second year the interest of the first annuity is rA (Art. 396); at the end of this year the principal becomes $2A$, therefore the interest due at the end of the third year is $2rA$; in the same manner, the interest due at the end of the fourth year is $3rA$; &c. Hence the whole interest at the end of n years is

$$rA + 2rA + 3rA \ldots\ldots + \overline{n-1} \,.\, rA = n \,.\, \frac{n-1}{2} rA \text{ (Art. 282);}$$

and the sum of the annuities is nA, therefore the whole amount

$$M = nA + n \,.\, \frac{n-1}{2} rA.$$

409. *Required the Present Value of an annuity which is to continue a certain number of years, allowing simple interest for the money.*

Let P be the present value; then if P, and the annuity, at the same rate of interest, amount to the same sum, they are upon the whole of equal value. The amount of P in n years is $P + Pnr$ (Art. 396); and the amount of the annuity in the same time is $nA + n \cdot \frac{n-1}{2} rA$; hence

$$P + Pnr = nA + n \cdot \frac{n-1}{2} rA;$$

$$\therefore\ P = \frac{nA + n \cdot \frac{n-1}{2} rA}{1 + nr}.$$

From this equation, any three of the four quantities P, A, n, r being given, the other may be found.

410. COR. Let n be infinite, then $P = \frac{nA}{2}$ an infinite quantity; therefore for a finite annuity to continue for ever, an infinite sum ought, according to this calculation, to be paid; a conclusion which shews the necessity of estimating the Present Value of an annuity upon different principles.

411. *To find the Amount of an annuity in any number of years, at compound interest.*

Let A be the annuity, or sum due at the end of the first year; then $1 : R :: A : RA$, its amount at the end of the second year; therefore $A + RA$ is the sum due at the end of the second year; in the same manner,

$$1 : R :: (1 + R) \times A : (R + R^2) \times A,$$

the amount of the two payments at the end of the third year; and $(1 + R + R^2) \times A$ is the whole sum due at the end of the third year; in the same manner,

$$(1 + R + R^2 \ldots + R^{n-1}) \times A$$

is the sum due at the end of n years, that is, the amount required

$$M = \frac{R^n - 1}{R - 1} \times A.$$

412. Cor. 1. From this equation, any three of the quantities being given, the fourth may be found.

$$A = M.\frac{R-1}{R^n-1},\quad n = \frac{\log\left(1+\frac{M}{A}.\overline{R-1}\right)}{\log R}.$$

Cor. 2. If interest be paid q times per annum, and $\frac{r}{q}$ be each payment per 1£, the amount of the annuity in n years, reckoning compound terest, will be

$$A \times \frac{\left(1+\frac{r}{q}\right)^{nq}-1}{\left(1+\frac{r}{q}\right)^{q}-1}.$$

Cor. 3. If the annuity (A) be payable m times per annum, each the payments being $\frac{A}{m}$, and ρ be the *annual* rate of interest, the nount in n years will be

$$\frac{A}{m}.\frac{(1+\rho)^n-1}{(1+\rho)^{\frac{1}{m}}-1};$$

d if the interest also be payable q times a year, each payment of inrest for every 1£ being $\frac{r}{q}$, this amount becomes

$$\frac{A}{m}.\frac{\left(1+\frac{r}{q}\right)^{nq}-1}{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1}.$$

413. *To find the Present Value of an annuity to be paid r* n *years, allowing compound interest.*

Let P be the Present Value, A the annuity; then since PR^n is e amount of P in n years, and $\frac{R^n-1}{R-1} \times A$ the amount of A in e same time, by the question,

$$PR^n = \frac{R^n-1}{R-1}\times A,\ \therefore\ P = \frac{1-R^{-n}}{R-1}\times A,\ \text{or} = A.\frac{1-(1+r)^{-n}}{r}.$$

414. Cor. 1. Any three of the quantities P, A, R, n being ven, the fourth may be found.

415. **Cor. 2.** If the number of years be infinite, R^n is infinite, and R^{-n} vanishes; therefore $P = \dfrac{A}{R-1}$, or $\dfrac{A}{r}$.

Ex. If the annual rent of a freehold estate be 1£, what is its value, allowing 5 per cent. per ann. compound interest?

In this case, $A = 1$, $R - 1 = 0{\cdot}05$; therefore the Present Value $P = \dfrac{1}{0{\cdot}05} = £20$, or 20 years purchase.

Cor. 3. The Present Value of an Annuity of A£ payable m times per annum for n years, each of the payments being $\dfrac{A}{m}$, and ρ the *annual* rate of interest, will be

$$\frac{A}{m}\cdot\frac{1-(1+\rho)^{-n}}{(1+\rho)^{\frac{1}{m}}-1};$$

and if the interest also be payable q times a year, each payment of interest for every £1 being $\dfrac{r}{q}$, the Present Value will be

$$\frac{A}{m}\cdot\frac{1-\left(1+\frac{r}{q}\right)^{-nq}}{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1}.$$

Cor. 4. If the annuity is to continue for ever, this Present Value becomes

$$\frac{A}{m}\cdot\frac{1}{\left(1+\frac{r}{q}\right)^{\frac{q}{m}}-1}$$

416. The Present Value of an annuity, to commence at the expiration of p years, and to continue q years, is the difference between its present value for $\overline{p+q}$ years, and its present value for p years,

$$= \frac{A}{R-1} - \frac{AR^{-\overline{p+q}}}{R-1} - \left\{\frac{A}{R-1} - \frac{AR^{-p}}{R-1}\right\}$$

$$= \frac{AR^{-p}}{R-1}\{1-R^{-q}\}.$$

Cor. If the annuity commences after p years, and continues for ever, the Present Value will be $\dfrac{AR^{-p}}{R-1}$.

Ex. What is the Present Value of an annuity of 1£, for 14 ears, to commence at the expiration of 7 years, allowing 5 per ent. per ann. compound interest?

$$\text{The Present Value for 21 years} = \frac{(1{\cdot}05)^{21} - 1}{(1{\cdot}05)^{21} \times 0{\cdot}05} = 12{\cdot}82\pounds;$$

$$\text{and the Present Value for 7 years} = \frac{(1{\cdot}05)^{7} - 1}{(1{\cdot}05)^{7} \times 0{\cdot}05} = 5{\cdot}79\pounds;$$

ence the value of the annuity for fourteen years after the expiraon of 7 is 7·03£, or 7 years purchase, nearly.

The preceding Article contains the whole Theory of the

RENEWAL OF LEASES.

417. *To determine the fine which ought to be paid for renewing any mber of years lapsed in a lease.*

Let $p + q$ be the number of years for which the lease was originally anted; p the number lapsed; and A the clear annual value of the tate, after deducting reserved rent (if any), taxes, and all other fixed nual charges.

Then it is clear that the lessee has to purchase an annuity of A£ commence at the expiration of q years, and to continue p years, the resent Value of which is the Present Value for $\overline{p+q}$ years – Present alue for q years

$$= \frac{A}{R-1} \cdot \{R^{-q} - R^{-(p+q)}\}.$$

Cor. The number of years purchase is

$$\frac{1}{R-1}\{R^{-q} - R^{-(p+q)}\}.$$

Ex. In a lease of 21 years 7 years lapsed are to be renewed, the eserved rent is 10£, and the estate is really worth 150£ a year, what fine ought to be paid for the renewal, reckoning interest at 5 per cent.?

In this case the lessee has to pay for an annuity of 140£ to commence at the end of 14 years and to continue 7 years; therefore the fine required is

$$\frac{140}{0{\cdot}05}\{\overline{1{\cdot}05}|^{-14} - \overline{1{\cdot}05}|^{-21}\}\ \pounds.$$

$$\text{Now } \frac{140}{0{\cdot}05} = 2800,$$

$$\log_{10} \overline{1{\cdot}05}|^{-14} = -\,0{\cdot}29666 = \bar{1}{\cdot}70334 = \log_{10} 0{\cdot}50505$$

$$\log_{10} \overline{1{\cdot}05}|^{-21} = -\,0{\cdot}44499 = \bar{1}{\cdot}55501 = \log_{10} 0{\cdot}35895$$

$$\therefore \text{ the required fine } = 2800 \times \{0{\cdot}50505 - 0{\cdot}35895\}$$
$$= 2800 \times 0{\cdot}1461$$
$$= 409£, \text{ very nearly.}$$

SCHOLIUM.

418. The method of determining the present value of an annuity at simple interest, given in Art. 409, has been decried by several eminent Arithmeticians, and in its stead a solution of the question has been proposed upon the following principle; "If the present value of each payment be determined separately, the sum of these values must be the value of the whole annuity".

Let x be the value or price paid down for the annuity, a the yearly payment, n the number of years for which it is to be paid, r the interest of 1£ for one year. The present value of the first payment is $\frac{a}{1+r}$ (Art. 397); the present value of the second payment, or of a£ to be paid at the end of two years, is $\frac{a}{1+2r}$; and so on: therefore $x = \frac{a}{1+r} + \frac{a}{1+2r} \ldots + \frac{a}{1+nr}$.

These different conclusions arise from a circumstance which the opponents seem not to have attended to. According to the former solution, no part of the interest of the price paid down is employed in paying the annuity, till the principal is exhausted.

Let the annuity be always paid out of the principal x as long as it lasts, and afterwards out of the interest which has accrued; then x, $x-a$, $x-2a$, $x-3a$, &c. are the sums in hand, during the first, second, third, fourth, &c. years, the interest arising from which rx, $rx-ra$, $rx-2ra$, $rx-3ra$, &c. that is, the whole interest, is $nrx - \{1+2+3\ldots(n-1)\} \times ra$, or, $nrx - n \cdot \frac{n-1}{2} ra$, which, together with the principal x, is equal to the sum of all the annuities; therefore

$$(1+nr)x - n \cdot \frac{n-1}{2} ra = na, \text{ and } x = \frac{na + n \cdot \frac{n-1}{2} ra}{1+nr} \text{ (Art. 409).}$$

According to the other calculation, part of the interest, as it arises, is employed in paying the annuity, but not the whole. Thus, the first payment is made by a part of the principal, and the interest of that part, which together amount to the annuity; and the other payments are made in the same manner; this is, in effect, allowing interest upon that part of the whole interest which is incorporated with the principal. According to either calculation, the seller has the advantage, since the whole or a part of the interest will remain at his disposal till the last annuity is paid off.

If the whole interest, as it arises, be incorporated with the principal, and employed in paying the annuity, *compound interest* is, in effect, allowed upon the whole. Let x be the price paid for the annuity, n the number of years for which it is granted, and $R = 1£$ together with its interest for one year. Then x in one year amounts to Rx, out of which the annuity being paid, $Rx - a$ is the sum in hand at the end of the first year; $R^2x - Ra$ is the amount of this sum at the end of the second year, therefore $R^2x - Ra - a$ is the sum in hand at the end of the second year; in the same manner, $R^nx - R^{n-1}a - R^{n-2}a \dots - a$ is the sum left, after paying the last annuity, which ought to be nothing; hence

$$R^n x = R^{n-1}a + R^{n-2}a + \dots + a = \frac{R^n a - a}{R - 1}.$$

$$\therefore\ x = \frac{(R^n - 1)a}{R^n \times (R - 1)}. \quad \text{(See Art. 413).}$$

CHANCES OR PROBABILITIES.

419. *Chance*, or *Probability*, has two meanings; the one a popular meaning, without any very distinct signification; the other a mathematical meaning, pointing out *a real value* existing in the circumstances.

Def. Most questions of probabilities will fall under one of two classes, called *direct* and *inverse* probabilities.

A question of probability is termed *direct*, when, certain causes being given as existent, from which a certain event may proceed, the probability of that event happening is required.

A question of probability is termed *inverse*, when, an event being given as existent, and proceeding from one of several causes, the probability of one proposed cause being the true one is required.

Some more complex questions may partake of the nature of both kinds of probability.

I. DIRECT PROBABILITIES.

420. *If an event may take place in* n *different ways, and each of these be equally likely to happen, the probability that it will take place in a specified way is properly represented by* $\frac{1}{n}$, *certainty being represented by* 1. *Or, which is the same thing, if the value of certainty be* 1, *the* value of the expectation *that the event will happen in a specified way is* $\frac{1}{n}$.

For the sum of all the probabilities is certainty, or 1, because the event must take place in some one of the ways; and the probabilities are equal: therefore each of them is $\frac{1}{n}$.

421. Cor. If the value of certainty be a, the value of the expectation is $\frac{a}{n}$. But in the following Articles we suppose the value of certainty to be 1.

422. *If an event may happen in* a *ways, and fail in* b *ways, any of these being equally probable, the chance of its happening is* $\frac{a}{a+b}$, *and the chance of its failing is* $\frac{b}{a+b}$.

The chance of its happening must, from the nature of the supposition, be to the chance of its failing, as $a : b$; therefore the chance of its happening : chance of its happening together with the chance of its failing :: $a : a+b$. And the event must either happen or fail; consequently the chance of its happening together with the chance of its failing is certainty. Hence the chance of its happening : certainty :: $a : a+b$; or the chance of its happening $= \frac{a}{a+b}$.

Also, since the chance of its happening together with the chance of its failing is certainty, which is represented by 1, $1 - \frac{a}{a+b}$, that is, $\frac{b}{a+b}$ is the chance of its failing.

423. Ex. 1. The probability of throwing an ace with a single die, in one trial, is $\frac{1}{6}$; the probability of not throwing an ace is $\frac{5}{6}$; the probability of throwing either an ace or a deuce is $\frac{2}{6}$; &c.

424. Ex. 2. If n balls, a, b, c, d, &c. be thrown promiscuously into a bag, and a person draw out one of them, the probability that it will be a is $\frac{1}{n}$; the probability that it will be either a or b is $\frac{2}{n}$.

425. Ex. 3. The same supposition being made, if two balls be drawn out, the probability that these will be a and b is $\frac{2}{n(n-1)}$.

For there are $n.\frac{n-1}{2}$ combinations of n things taken two and two together (Art. 300); and each of these is equally likely to be taken; therefore the probability that a and b will be taken is $\frac{1}{n.\frac{n-1}{2}}$, or $\frac{2}{n(n-1)}$.

426. Ex. 4. If 6 white and 5 black balls be thrown promiscuously into a bag, and a person draw out one of them, the probability that this will be a white ball is $\frac{6}{11}$; and the probability that it will be a black ball is $\frac{5}{11}$.

From the Bills of Mortality in different places Tables* have been constructed which shew how many persons, upon an average, out of a certain number born, are left at the end of each year, to the extremity of life. From such Tables the probability of the continuance of a life, of any proposed age, is known.

427. Ex. 1. *To find the probability that an individual of a given age will live one year.*

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* Some of these Tables will be found at the end of the Section.

Let A be the number, in the Tables, of the given age, B the number left at the end of the year; then $\frac{B}{A}$ is the probability that the individual will live one year; and $\frac{A-B}{A}$ the probability that he will die in that time (Art. 422). In Dr. Halley's Tables, out of 586 of the age of 22, 579 arrive at the age of 23; hence the probability that an individual aged 22 will live one year is $\frac{579}{586}$ or $\frac{1}{1+\frac{7}{579}}$ or $\frac{1}{1+\frac{1}{82+\frac{5}{7}}}$ or $\frac{82}{83}$ nearly; and $\frac{7}{586}$, or $\frac{1}{84}$ nearly, is the probability that he will die in that time.

428. Ex. 2. *To find the probability that an individual of a given age will live any number of years.*

Let A be the number in the Tables of the given age; $B, C, D, \ldots X$, the number left at the end of 1, 2, 3, ...x, years; then $\frac{B}{A}$ is the probability that the individual will live 1 year; $\frac{C}{A}$ the probability that he will live 2 years; and $\frac{X}{A}$ the probability that he will live x years. Also $\frac{A-B}{A}, \frac{A-C}{A}, \frac{A-X}{A}$, are the probabilities that he will die in 1, 2, x years, respectively.

These conclusions follow immediately from Art. 422.

429. *If two events be independent of each other, and the probability that one will happen be* $\frac{1}{m}$, *and the probability that the other will happen be* $\frac{1}{n}$, *the probability that they will both happen is* $\frac{1}{mn}$; m *and* n *being the numbers of ways in which the events can severally happen or fail.*

For each of the *m* ways, in which the first can happen or fail, may be combined with each of the *n* ways in which the other can happen or fail, and thus form *mn* combinations; and there is only one in which both can happen; therefore the probability that this will be the case is $\frac{1}{mn}$ (Art. 420).

430. Cor. 1. The probability that both do not happen is $1-\frac{1}{mn}$, or $\frac{mn-1}{mn}$. For the probability that they both happen, together with the probability that they do not both happen, is certainty; therefore, if from 1 the probability that they both happen be subtracted, the remainder is the probability that they do not both happen.

431. Cor. 2. The probability that they will both fail is $\frac{(m-1)(n-1)}{mn}$. For the probability that the first will fail is $\frac{m-1}{n}$, and the probability that the second will fail is $\frac{n-1}{n}$; therefore the probability that they will both fail is $\frac{m-1}{m}\times\frac{n-1}{n}$, or $\frac{(m-1)(n-1)}{mn}$.

432. Cor. 3. The probability that one will happen and the other fail is $\frac{m+n-2}{mn}$. For the first may happen and the second fail in $n-1$ ways; the second may happen and the first fail in $m-1$ ways. Hence the number of favourable cases is $m+n-2$, whilst the number of favourable and unfavourable cases together is *mn*. Therefore the probability required is $\frac{m+n-2}{mn}$.

433. Cor. 4. If there be any number of independent events, and the probabilities of their happening be $\frac{1}{m}$, $\frac{1}{n}$, $\frac{1}{r}$, &c. respectively, the probability that they will all happen is $\frac{1}{mnr\ \&c.}$. For the probability that the first two will happen is $\frac{1}{mn}$, and the pro-

bability that the first two and third will happen is $\frac{1}{mnr}$; and the same proof may be extended to any number of events.

When $m = n = r =$ &c. the probability is $\frac{1}{m}$, v being the number of events.

434. Ex. 1. Required the probability of throwing an ace and then a deuce with one die.

The chance of throwing an ace is $\frac{1}{6}$, and the chance of throwing a deuce in the second trial is $\frac{1}{6}$; therefore the chance of both happening is $\frac{1}{36}$.

435. Ex. 2. If 6 white and 5 black balls be thrown promiscuously into a bag, what is the probability that a person will draw out first a white and then a black ball?

The probability of drawing a white ball first is $\frac{6}{11}$ (Art. 426), and this being done, the probability of drawing a black ball is $\frac{5}{10}$, or $\frac{1}{2}$, because there are 5 white and 5 black balls left; therefore the probability required is $\frac{6}{11} \times \frac{1}{2}$ or $\frac{3}{11}$.

Or we may reason thus; unless the person draw a white ball first, the whole is at an end; therefore the probability that he will have a chance of drawing a black ball is $\frac{6}{11}$, and when he has this chance, the probability of its succeeding is $\frac{5}{10}$, or $\frac{1}{2}$; therefore, the probability that both these events will take place is $\frac{6}{11} \times \frac{1}{2}$ or $\frac{3}{11}$.

436. Ex. 3. The same supposition being made as in the last example, what is the chance of drawing a white ball and then two black balls?

The probability of drawing a white ball and then a black one is $\frac{3}{11}$ (Art. 435); when these two are removed, there are 5 white and 4 black balls left; and the probability of drawing a black ball, out of these, is $\frac{4}{9}$; therefore the probability required is $\frac{3}{11} \times \frac{4}{9}$ or $\frac{4}{33}$.

437. Ex. 4. Required the probability of throwing an ace, with a single die, once at least, in two trials.

The chance of failing the first time is $\frac{5}{6}$, and the chance of failing the next is $\frac{5}{6}$; therefore the chance of failing twice together is $\frac{25}{36}$; and the chance of not failing both times is $1 - \frac{25}{36}$, or $\frac{11}{36}$.

438. Ex. 5. In how many trials may a person undertake, for an even wager, to throw an ace with a single die?

Let x be the number of trials; then, as in the last Art., the chance of failing x times together is $\left(\frac{5}{6}\right)^x$, and this, by the question, is equal to the chance of happening, or

$$\left(\frac{5}{6}\right)^x = \frac{1}{2};$$

$$\text{hence} \quad x \times \log\frac{5}{6} = \log\frac{1}{2};$$

$$\text{or} \quad x \times (\log 5 - \log 6) = \log 1 - \log 2,$$

$$x = \frac{\log 1 - \log 2}{\log 5 - \log 6} = \frac{\log 2}{\log 6 - \log 5}, \quad \text{since } \log 1 = 0;$$

$$\therefore \; x = 3{\cdot}8, \text{ nearly};$$

that is, a person would safely undertake to throw the ace in four trials, but not less, and then have some probability to spare in his favour.

439. Ex. 6. *To find the probability that two individuals* P *and* Q, *whose ages are known, will live a year.*

Let the probability that P will live a year, determined by Art. 427, be $\frac{1}{m}$; and the probability that Q will live a year be $\frac{1}{n}$; then the probability that they will both be alive at the end of that time is $\frac{1}{m}\times\frac{1}{n}$ or $\frac{1}{mn}$.

440. Ex. 7. *To find the probability that one of them, at least, will be alive at the end of any number of years.*

The probability that P will die in a year is $\frac{m-1}{m}$, and the probability that Q will die is $\frac{n-1}{n}$; therefore the probability that they will both die is $\frac{(m-1)(n-1)}{mn}$; and the probability that they will not both die is

$$1-\frac{(m-1)(n-1)}{mn}, \text{ or } \frac{m+n-1}{mn}.$$

In the same manner, if $\frac{1}{p}$ be the probability that P will live t years, and $\frac{1}{q}$ the probability that Q will live the same time (Art. 428), the probability that one of them, at least, will be alive at the end of the time is

$$1-\frac{(p-1)(q-1)}{pq}, \text{ or } \frac{p+q-1}{pq}.$$

441. *If the probability of an event's happening in one trial be represented by* $\frac{\mathrm{a}}{\mathrm{a}+\mathrm{b}}$ (Art. 422), *to find the probability of its happening once, twice, three times, &c. exactly, in* n *trials.*

The probability of its happening in any one particular trial being $\frac{a}{a+b}$, the probability of its failing in all the other $n-1$ trials is $\frac{b^{n-1}}{(a+b)^{n-1}}$ (Arts. 422, 433); therefore the probability of its appening in one particular trial, and failing in the rest, is $\frac{ab^{n-1}}{(a+b)^n}$; d since there are n trials, the probability that it will happen in me one of these, and fail in the rest, is n times as great, or $\frac{ab^{n-1}}{+ b)^n}$.

The probability of its happening in any two particular trials, d failing in all the rest, is $\frac{a^2b^{n-2}}{(a+b)^n}$, and there are $n \cdot \frac{n-1}{2}$ ays in which it may happen twice in n trials and fail in all the st (Art. 300); therefore the probability that it will happen twice n trials is $\frac{n \cdot \frac{n-1}{2} a^2b^{n-2}}{(a+b)^n}$.

In the same manner, the probability of its happening exactly ree times is $\frac{n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} a^3b^{n-3}}{(a+b)^n}$; and the probability of its appening exactly t times is

$$\frac{n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \dots\dots \frac{n-t+1}{t} a^tb^{n-t}}{(a+b)^n}.$$

442. Cor. 1. The probability of the event's failing exactly times in n trials may be shewn, in the same way, to be

$$\frac{n \cdot \frac{n-1}{2} \cdot \frac{n-2}{3} \dots\dots \frac{n-t+1}{t} a^{n-t}b^t}{(a+b)^n}.$$

443. Cor. 2. The probability of the event's happening *at east* t times in n trials is

$$\frac{a^n + na^{n-1}b + n \cdot \frac{n-1}{2} a^{n-2}b^2 + \dots\dots \text{ to } (n-t+1) \text{ terms}}{(a+b)^n}.$$

For, if it happen every time, or fail only once, twice, ... $n-t$) times, it happens t times; therefore the whole probability of its happening, at least t times, is the sum of the probabilities of its happening every time, of failing only once, twice, ... $n-t$ times; and the sum of these probabilities is

$$\frac{a^n + na^{n-1}b + n\,.\,\frac{n-1}{2}\,a^{n-2}b^2 + \ldots\ldots \text{ to } (n-t+1) \text{ terms}}{(a+b)^n}.$$

444. Ex. 1. What is the probability of throwing an ace twice at least, in three trials, with a single die?

In this case $n = 3$, $t = 2$, $a = 1$, $b = 5$; and the probability required is $\frac{1 + 3 \times 5}{6 \times 6 \times 6} = \frac{16}{216} = \frac{2}{27}$.

445. Ex. 2. What is the probability that out of five individuals, of a given age, three at least will die in a given time?

Let $\frac{1}{m}$ be the probability that any one of them will die in the given time (Art. 428); then we have given the probability of an event's happening in one instance, to find the probability of its happening three times, at least, in five instances.

In this case $a = 1$, $b = m - 1$, $n = 5$, $t = 3$; therefore the probability required is

$$\frac{1 + 5(m-1) + 10(m-1)^2}{m^5}.$$

II. INVERSE PROBABILITIES.

446. Axiom. When an event can proceed from one of a system of causes, the probabilities of these causes having produced the event are proportional to the numbers of ways in which they can severally produce the event.

447. Cor. Hence the probabilities of the several causes having produced the event are proportional to the chances of the event happening on the assumption of their being severally existent.

If p_1, p_2, p_3, &c. be the probabilities of several causes from which an event may proceed; a_1, a_2, a_3, &c. the chances of that event happening on supposition of these causes severally existing; we have

$$\frac{p_1}{a_1} = \frac{p_2}{a_2} = \frac{p_3}{a_3} = \text{\&c.}$$

$$\therefore \frac{p_1}{a_1} = \frac{\Sigma(p)}{\Sigma(a)}.$$

But as *some one* of the system of causes is known to be the true e, $\Sigma(p)$ is certainty, or 1,

$$\therefore p_1 = \frac{a_1}{\Sigma(a)}.$$

$$\text{So also } p_2 = \frac{a_2}{\Sigma(a)}, \quad p_3 = \frac{a_3}{\Sigma(a)}, \quad \text{\&c.}$$

Ex. An urn contains 3 balls which may be white or black. A ll is drawn out and replaced three times, and in each case a white ll is drawn. What are the probabilities of the urn containing (1) Three iite balls: (2) Two white and one black: (3) One white and two ick: (4) Three black?

(1) Supposing the 1st state of the urn, the chance of the event hapning which did happen is $\frac{3}{3} \times \frac{3}{3} \times \frac{3}{3}$ or $\frac{27}{27}$, that is, certainty.

(2) Supposing the 2nd state, the chance is $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$, or $\frac{8}{27}$.

(3) Supposing the 3rd state, the chance is $\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3}$, or $\frac{1}{27}$.

(4) Supposing the 4th state, the chance is $\frac{0}{3} \times \frac{0}{3} \times \frac{0}{3}$, or 0.

$\therefore$ the chance *for* the 1st state of the urn is $\frac{27}{27} \div \frac{27+8+1}{27}$, or $\frac{27}{36}$.

The chance *for* the 2nd is $\frac{8}{27} \div \frac{36}{27}$, or $\frac{8}{36}$.

.... 3rd is $\frac{1}{27} \div \frac{36}{27}$, or $\frac{1}{36}$.

.... 4th is $\frac{0}{27} \div \frac{36}{27}$, or 0.

The following Proposition involves the nature of both *direct and iverse* probabilities:—

448. *If* a_1, a_2, a_3, *&c. be the chances of an observed event on supposition of each of the system of causes being the true one;* a_1', a_2', a_3', *&c. e probabilities of another proposed event on the same separate supposions; the chance of this event happening* $= \frac{\Sigma(aa')}{\Sigma(a)}$.

For, as has been seen, a_1, a_2, a_3, &c. are proportional to the chances iven by the observed event of the separate causes existing. Hence the umber of ways of the first cause existing and the second event happening om it is proportional to $a_1 a_1'$; and the number of ways of that event ippening from some one of the causes is proportional to $\Sigma(aa')$. But nce some one of the causes exists, $\Sigma(a)$ is in the same proportion to the number of ways in which one of the causes may exist, and the event either happen or fail from it. Hence the chance of the event happening $=\frac{\Sigma(aa')}{\Sigma(a)}$.

Ex. 1. An urn contains two balls, but whether white or black, uncertain—we draw one ball, and find it is white. The ball is then replaced; what is the chance of next drawing a black one?

Before the drawing takes place the two states or causes may be, (1) Two white, and (2) One white and one black.

Now the chance of the *observed* event under (1) is $\frac{2}{2}$, or 1,

.... (2) is $\frac{1}{2}$.

Again, the chance of the *proposed* event under (1) is 0,

.... (2) is $\frac{1}{2}$.

$$\therefore \text{ the Chance required } = \frac{1\times 0+\frac{1}{2}\times\frac{1}{2}}{1+\frac{1}{2}}=\frac{1}{6}.$$

Ex. 2. Taking the case supposed in Ex. Art. 447, what is the chance of a white ball coming out at a fourth such drawing?

Here before the drawing takes place the states or causes may be, (1) Three white, (2) Two white and one black, (3) One white and two black.

Now the chance of the *observed* event under (1) is 1;

.... (2) is $\frac{8}{27}$,

.... (3) is $\frac{1}{27}$,

Again, the chance of the *proposed* event under (1) is 1;

.... (2) is $\frac{2}{3}$,

.... (3) is $\frac{1}{3}$,

$$\therefore \text{ the Chance required } = \frac{1\times 1+\frac{8}{27}\times\frac{2}{3}+\frac{1}{27}\times\frac{1}{3}}{1+\frac{8}{27}+\frac{1}{27}}=\frac{98}{108}=\frac{49}{54}.$$

SCHOLIUM.

Much more might be said on a subject so extensive as the doctrine of Chances; the Learner will however find the principal grounds of calculation in Articles 420, 422, 429, 441, 443, 447, and 448; and if he wish for farther information, he may consult De Moivre's work on this subject*. It may not be improper to caution him against applying principles which, on the first view, may appear self-evident; as there is no subject in which he will be so likely to mistake as in the calculation of probabilities. A single instance will shew the danger of forming a hasty judgment, even in the most simple case. The probability of throwing an ace with one die is $\frac{1}{6}$, and since there is an equal probability of throwing an ace in the second trial, it might be supposed that the probability of throwing an ace in two trials is $\frac{2}{6}$.

This is not a just conclusion (Art. 437); for, it would follow, by the same mode of reasoning, that in six trials a person could not fail to throw an ace. The error, which is not easily seen, arises from a tacit supposition that there must necessarily be a second trial, which is not the case if an ace be thrown in the first.

LIFE ANNUITIES.

449. *To find the Present Value of an annuity of £1 to be continued during the life of an individual of a given age, allowing compound interest for the money.*

Let R be the amount of £1 in one year; A the number of persons, in the Tables, of the given age; B, C, D, &c. the number left at the end of 1, 2, 3, &c. years; then $\frac{B}{A}$ is the value of the life for one year, $\frac{C}{A}$, $\frac{D}{A}$, &c. its value for 2, 3, &c. years respectively; and the series must be continued to the end of the Tables. Now the Present Value of £1, to be paid at the end of one year, is $\frac{1}{R}$ (Art. 397); but it is only to be paid on condition that the annuitant is alive at the end of the year, of which event the probability is $\frac{B}{A}$; therefore the Present Value of the conditional annuity is $\frac{B}{AR}$ (Art. 421); in the same manner, the Present Value of the second year's annuity is $\frac{C}{AR^2}$; the Present Value of the third year's annuity is $\frac{D}{AR^3}$, &c.; therefore the whole value required is $\frac{1}{A} \times \left(\frac{B}{R} + \frac{C}{R^2} + \frac{D}{R^3} + \text{\&c. to the end of the Tables.}\right)$

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* The more modern writers on this subject are Laplace, Galloway, and De Morgan.

450. De Moivre supposes, that out of eighty-six persons born one dies every year, till they are all extinct.

This supposition is sufficiently exact, if our calculations be made for any age, neither very young nor very old, as will appear from an inspection of the Tables; and, on this supposition, the sum of the series $\frac{1}{A} \times \left(\frac{B}{R} + \frac{C}{R^2} + \frac{D}{R^3} + \text{\&c.}\right)$ may be found.

Let n be the number of years which any individual wants of 86; then will n be the number of persons living, of that age, out of which one dies every year; and

$$\frac{n-1}{n},\ \frac{n-2}{n},\ \frac{n-3}{n},\ \text{\&c.}$$

will be the probabilities of his living 1, 2, 3, &c. years; hence, the Present Value of an annuity of £1, to be paid during his life, is

$$\frac{n-1}{nR} + \frac{n-2}{nR^2} + \frac{n-3}{nR^3} + \text{\&c. continued to } n \text{ terms.}$$

The sum of the series

$\frac{(n-1)x}{n} + \frac{(n-2)x^2}{n} + \frac{(n-3)x^3}{n} + \text{\&c.}$ to n terms is found to be (see Art. 295. Ex. 3) $\frac{(n-1)x - nx^2 + x^{n+1}}{n(1-x)^2}$;

let $x = \frac{1}{R}$, and the sum of the series

$$\frac{n-1}{nR} + \frac{n-2}{nR^2} + \frac{n-3}{nR^3} + \text{\&c. to } n \text{ terms} = \frac{(n-1)R - n + R^{-(n-1)}}{n(R-1)^2};$$

the Present Value of the annuity.

451. Cor. 1. This expression for the sum is the same with

$$\frac{nR-n}{n(R-1)^2}-\frac{R-R^{-(n-1)}}{n(R-1)^2}, \text{ or } \frac{1}{R-1}-\frac{R}{n}\times\frac{1-R^{-n}}{(R-1)^2}.$$

Let P be the Present Value of an annuity of £1, to continue cer-in for n years, then $P=\dfrac{1-R^{-n}}{R-1}$ (Art. 413); and the expression ecomes $\dfrac{1-\dfrac{R}{n}.P}{R-1}$

452. Cor. 2. The Present Value of the annuity to continue r ever, from the death of the proposed individual, is $\dfrac{RP}{n(R-1)}$.

For the whole Present Value of the annuity, to continue for er, is $\dfrac{1}{R-1}$ (Art. 415); and if from this its value for the life of e individual be taken, the remainder $\dfrac{RP}{n(R-1)}$ is the Present alue of the annuity, to continue for ever, from the time of his eath.

453. *To find the Present Value of an annuity of £1. to be id as long as two specified individuals are both living.*

Find, by Art. 439, the probability that they will both be alive the expiration of 1, 2, 3, &c. years, to the end of the Tables; ll these probabilities a, b, c, &c. and R the amount of £1, in one ear; then $\dfrac{a}{R}+\dfrac{b}{R^2}+\dfrac{c}{R^3}+$ &c. is the Present Value of the annuity quired. (See Art. 449).

454. *To find the Present Value of an annuity of £1, to be id as long as either of two specified individuals is living.*

Find, by Art. 440, the probability that they will not both be tinct in 1, 2, 3, &c. years, to the end of the Tables, and call these robabilities A, B, C, &c. then the Present Value of the annuity is

$$\frac{A}{R}+\frac{B}{R^2}+\frac{C}{R^3}+\text{ \&c. (See Art. 449).}$$

455. Cor. If the annuity be M£, the Present Value is M times as great as in the former case, or

$$M \times \left(\frac{A}{R} + \frac{B}{R^2} + \frac{C}{R^3} + \&c.\right)$$

456. These are the mathematical principles on which the values of annuities for lives are calculated, and the reasoning may easily be applied to every proposed case. But, in practice, these calculations, as they require the combination of every year of each life with the corresponding years of every other life concerned in the question, will be found extremely laborious, and other methods must be adopted when expedition is required. Writers on this subject are De Moivre, Maseres, Simpson, Price, Morgan, and Waring.

Other writers on the subject are Milne, Baily, and De Morgan, of which the last mentioned is now most accessible.

457. *To find the present value of the next presentation to a Living.*

Let i£ be the average annual *net* income of the living; c£ the cost of a curate, that is, the money value of the work to be done, and a£ the unavoidable expences of the admission of a new incumbent. Then the Present Value of the next presentation will obviously be the present value of an annuity of $(i-c)$£, to commence at the death of the present incumbent, and to continue during a life *then* 24 years of age, deducting the present value of a£ payable on admission.

Let n be the number of years which the Tables give to the present incumbent, p the number for a person 24 years of age; then the Present Value required will be that of an annuity of $(i-c)$£ to commence at the expiration of n years, and to continue p years, deducting the Present Value of a£ to be paid after n years*,

$$= \frac{i-c}{R-1} \cdot \left\{ R^{-n} - R^{-\overline{n+p}} \right\} - aR^{-n}.$$

Cor. The Present Value of an *Advowson*, or perpetual nomination to a living, will be that of an Annuity of $(i-c)$£ commencing after n years, and continuing for ever, deducting the present value of a£, to be paid at the end of n years, and also of the same sum to be paid at intervals of p years for ever afterwards,

$$= \frac{i-c}{R-1} \cdot R^{-n} - aR^{-n} - aR^{-\overline{n+p}} - aR^{-\overline{n+2p}} - \ldots\ldots\ldots in\ inf.$$

$$= \frac{i-c}{R-1} \cdot R^{-n} - aR^{-n} \cdot \frac{1}{1-R^{-p}}.$$

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* Of course this is on the supposition, that the laws, which permit such traffic in spiritual cures, remain in force, and that the values of i, c, and a remain unaltered, for $\overline{n+p}$ years at least.

TABLE I.

For determining the Probabilities of the Duration of Life, from Observations on the Bills of Mortality of **Breslaw**, made in the years 1687...1691, by Dr. Halley.

Age.	Persons living.	Decr. of Life.	Age.	Persons living.	Decr. of Life.	Age.	Persons living.	Decr. of Life.
1	1000	145	31	523	8	61	232	10
2	855	57	32	515	8	62	222	10
3	798	38	33	507	8	63	212	10
4	760	28	34	499	9	64	202	10
5	732	22	35	490	9	65	192	10
6	710	18	36	481	9	66	182	10
7	692	12	37	472	9	67	172	10
8	680	10	38	463	9	68	162	10
9	670	9	39	454	9	69	152	10
10	661	8	40	445	9	70	142	11
11	653	7	41	436	9	71	131	11
12	646	6	42	427	10	72	120	11
13	640	6	43	417	10	73	109	11
14	634	6	44	407	10	74	98	10
15	628	6	45	397	10	75	88	10
16	622	6	46	387	10	76	78	10
17	616	6	47	377	10	77	68	10
18	610	6	48	367	10	78	58	9
19	604	6	49	357	11	79	49	8
20	598	6	50	346	11	80	41	7
21	592	6	51	335	11	81	34	6
22	586	7	52	324	11	82	28	5
23	579	6	53	313	11	83	23	4
24	573	6	54	302	10	84	19	4
25	567	7	55	292	10	85	15	4
26	560	7	56	282	10	86	11	3
27	553	7	57	272	10	87	8	3
28	546	7	58	262	10	88	5	2
29	539	8	59	252	10	89	3	2
30	531	8	60	242	10	90	1	1

TABLE II.

For determining the Probabilities of Life at Northampton, as deduced by Dr. Price from the mortality of that town in the years 1741...1780.

Age.	Persons living.	Decr. of Life.	Age.	Persons living.	Decr. of Life.	Age.	Persons living.	Decr. of Life.
0	1149	300	31	428	7	62	187	8
1	849	127	32	421	7	63	179	8
2	722	50	33	414	7	64	171	8
3	672	26	34	407	7	65	163	8
4	646	21	35	400	7	66	155	8
5	625	16	36	393	7	67	147	8
6	609	13	37	386	7	68	139	8
7	596	10	38	379	7	69	131	8
8	586	9	39	372	7	70	123	8
9	577	7	40	365	8	71	115	8
10	570	6	41	357	8	72	107	8
11	564	6	42	349	8	73	99	8
12	558	5	43	341	8	74	91	8
13	553	5	44	333	8	75	83	8
14	548	5	45	325	8	76	75	8
15	543	5	46	317	8	77	67	7
16	538	5	47	309	8	78	60	7
17	533	5	48	301	8	79	53	7
18	528	6	49	293	9	80	46	7
19	522	7	50	284	9	81	39	7
20	515	8	51	275	8	82	32	6
21	507	8	52	267	8	83	26	5
22	499	8	53	259	8	84	21	4
23	491	8	54	251	8	85	17	4
24	483	8	55	243	8	86	13	3
25	475	8	56	235	8	87	10	2
26	467	8	57	227	8	88	8	2
27	459	8	58	219	8	89	6	2
28	451	8	59	211	8	90	4	2
29	443	8	60	203	8	91	2	1
30	435	7	61	195	8	92	1	1

The preceding Tables require but little explanation. The former commences by stating that out of 1000 persons who were born at the same time and attain the age of 1 year, 145 die before they attain the age of 2 years. Consequently at 2 years of age there are left 855 out of 1000.

these 57 die between 2 and 3 years of age; and so on. Thus, of 1000 sons who attain the age of one year, the Table indicates that 346 live e 50 years of age; &c.

The latter Table commences a year earlier by taking 1149 persons n together, that is, at the age 0; and then proceeds in the same nner as the former. Thus, we have given by this Table, that after years, out of 1149 persons born together, 284 are then alive, and that hese 9 die before attaining the age of 51; and so on.

It has been objected to both the preceding Tables, although the er is very generally used by the Assurance offices, that they make no inction between male and female life, and yet that a very material inction can be proved to exist.

TABLE III.

wing the Expectation of Life, as deduced by Professor De Morgan from the Statistical returns of the whole of Belgium made by M. Quetelet and Smits.

Age.	Towns.		Both.	Country.		Age.
	Males.	Females.	Both.	Males.	Females.	
0	29·2	33·3	32·2	32·0	32·9	0
5	45·0	47·1	45·7	46·1	44·8	5
10	42·9	45·0	43·9	44·4	42·9	10
15	39·0	41·3	40·5	41·2	40·0	15
20	35·4	38·0	37·3	38·1	37·0	20
25	33·1	35·0	34·7	35·7	34·2	25
30	30·4	32·1	32·0	33·0	31·5	30
35	27·5	29·2	28·9	29·7	28·7	35
40	24·4	26·5	25·8	26·0	25·9	40
45	21·5	23·3	22·7	22·5	23·2	45
50	18·3	20·1	19·5	19·1	20·0	50
55	15·5	17·1	16·4	16·2	16·9	55
60	12·8	14·0	13·4	13·3	13·7	60
65	10·4	11·2	10·8	10·6	10·9	65
70	8·2	8·6	8·4	8·2	8·5	70
75	6·3	6·6	6·4	6·3	6·5	75
80	4·8	5·1	5·0	5·0	5·1	80
85	3·7	4·0	3·8	3·8	3·8	85
90	2·9	3·0	3·1	3·1	3·2	90
95	1·8	2·0	2·1	2·2	1·9	95
100	0·0	0·5	1·3	0·5	0·5	100

The extent of the error which arises from not distinguishing between the sexes may be seen in Table III. constructed by Professor De Morgan from the statistical returns of *the whole of Belgium* for three successive years, as given by M. Quetelet and Smits, in the *Recherches sur la Reproduction, &c.* Brussels, 1832. This Table is calculated to shew the "expectation of life," that is, the average number of years remaining to any individual, at intervals of five years, from the age of 0 to 100. It distinguishes not only between male and female, but between town life and rural life; and the middle column gives the general average for the whole kingdom, male and female, town and country.

DISCUSSION AND INTERPRETATION OF ANOMALOUS RESULTS.

458. *Negative Results.* It often happens, that the result of our operations for the solution of a proposed question or problem appears in a negative form, although, strictly speaking, there can be no such thing existent as an essentially negative quantity. But it will always be found, when such a result occurs, that there is something in the nature of the question which will either dispose of, or supply a meaning to, the negative result. Thus, to take a simple example; suppose a man wishes to ascertain the amount of his property—he puts down what he has, together with what is due to him, as *positive,* and all his debts with a *negative* sign. If then he finds that by taking the sum of both positive and negative quantities, the result is *negative,* its meaning will be sufficiently obvious, viz. that his property is so much less than nothing, that is, he is so much in debt. See Scholium, p. 44; and Art. 214.

Also, see Art. 282. Ex. 4. In this and like cases it is true that two solutions may be found for the *equation,* that is, two values of n; but when either of those values is *fractional* or *negative,* it is clearly inadmissible as *a solution of the question proposed.*

It may be observed also here generally, that when in solving a *problem,* expressed algebraically, we find it necessary, as in the above Example, to extract the square root of a quantity, the double sign ±, (that is, + *or* −), need not to be prefixed to the root, at least for the object before us, if we have sufficient *data* beforehand for determining *which* sign the problem requires. Is it to be wondered at, that we produce an anomalous and unintelligible result, if we wilfully make a quantity negative which we know to be positive, or *vice versâ?*

Oftentimes, however, the negative solution, whether it results from carelessness or necessity, will satisfy *another problem* cognate with the proposed one; which may be determined by substituting the negative quantity for the positive in that step of the process which most clearly expresses the conditions of the question; and then interpreting the resulting equation with the assistance of the given problem. This is done in Art. 214.

Ex. The sum of n terms of the series $a, a+b, a+2b$, &c., *obtained on the supposition that* n *is a positive integer,* is $(2a+\overline{n-1}.b)\frac{n}{2}$; find the series of which this is the sum, supposing n a negative integer.

If n be negative, the expression for the sum becomes $(-2a+\overline{n+1}.b)\frac{n}{2}$; and to find the series, let $n=1$, then the sum $=b-a=$ the first term; again, let $n=2$, then the sum of two terms $=3b-2a$; $\therefore$ the second term $=3b-2a-(b-a)=2b-a$; and $\therefore$ the common difference $=b$, as in the original series. Hence the required series is

$$b-a,\quad 2b-a,\quad 3b-a,\quad 4b-a,\ \text{\&c.}$$

459. *Interminable Quotients.* Strictly speaking a *Quotient* can only exist when after the division by which it is determined there is no *Remainder;* but the term is applied to those cases also where a remainder is left which cannot be got rid of. Thus we say generally, that the quotient of $1 \div \overline{1-x}$ is $1+x+x^2+x^3+\ldots\ldots$; whereas the true quotient is $1+x+x^2+x^3+\ldots\ldots+x^{r-1}+\frac{x^r}{1-x}$. Thus, *whatever be the value of* x,

$$\frac{1}{1-x}=1+x+x^2+x^3+\ldots\ldots+x^{r-1}+\frac{x^r}{1-x}$$

$$=1+\frac{x}{1-x},\ \text{or}\ 1+x+\frac{x^2}{1-x},\ \text{or}\ 1+x+x^2+\frac{x^3}{1-x};\ \text{and so on.}$$

N. B. By taking the *Remainder* into account no unintelligible result can arise from substituting any particular value for x.

Cor. 1. If $x<1$, then the *Remainder* may be neglected, if a sufficient number of terms of the series are taken. (Art. 290. Cor. 2.)

If $x=1$, then $\frac{1}{0}=1+1+1+1+$&c. *in inf.* = an infinitely great number.

Cor. 2. If x be negative, we have $\frac{1}{1+x}=1-x+x^2-x^3+$&c. *in inf.*; in which if we put 1 for x, we get $\frac{1}{2}=1-1+1-1+$&c. $=0$, or 1, acccording as an even or an odd number of terms is taken; both of which results are obviously impossible.

Now, taking the *Remainder* into account, we have

$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots\ldots(-x)^r+\frac{(-x)^{r+1}}{1+x},$$

$$\text{and}\ \frac{1}{2}=1-1+1-1+\ldots\ldots\ldots\ldots\ldots\pm\frac{1}{2}=\frac{1}{2},\ \text{as it ought.}$$

Again, as it was shewn in Art. 323,

$$\frac{1}{(1-x)^2}=1+2x+3x^2+\ldots\ldots+rx^{r-1}+\frac{(r+1)x^r-rx^{r+1}}{(1-x)^2}$$

without which fractional *Remainder* no arithmetical equality subsists between the series and $\frac{1}{(1-x)^2}$. And it may be observed generally, that no equality subsists, for purposes of calculation, betwixt any infinite series without the "*Remainder,*" and the primitive quantity from which it was derived, unless the series is *convergent*, so that we can make the Remainder after r terms, by increasing r, as small as we please.

460. *To explain generally the results which assume the forms* $a \times 0$, $\frac{0}{a}$, a^0, 0^0, $\frac{a}{0}$, $\frac{0}{0}$.

(1). Since $a \times b$ signifies a taken b times, it is clear that, if 0 is to follow the same rule as other multipliers, $a \times 0$ signifies a taken 0 times, and is therefore equal to 0.

(2). Since $\frac{b}{a}$ expresses the number of times a is contained in b, $\frac{0}{a}$ will signify the number of times a is contained in 0, that is, 0 times, or $\frac{0}{a} = 0$.

(3). By the general Rule of Indices, $a^m \times a^n = a^{m+n}$, and $a^m \div a^n = a^{m-n}$. Now in the first of these let $m = 0$, then $a^0 . a^n = a^n$, if the same rule holds when one of the indices is 0; therefore a^0, as far as regards the rule for multiplication of powers, is equivalent to 1, or $a^0 = 1$.

Also, since $\frac{a^m}{a^m} = a^{m-m}$, $1 = a^0$, if the rule for division of powers holds when the powers are equal; therefore it also accords with the general rule for division that $a^0 = 1$.

Hence a^0, b^0, c^0, &c. are separately equal to 1, if 0 be admitted as an index subject to the same rule as other indices.

(4). Since it has been already explained, that *any* quantity raised to a power represented by 0 may be safely expressed by 1, it follows that $(a-b)^0 = 1$, whatever a and b may be. If then $a = b$, we have $0^0 = 1$.

(5). When an algebraical quantity is made to assume the form $\frac{a}{0}$, it is said to be infinitely great, and its value is expressed by the symbol ∞. All that is meant is, that if the denominator be made less than any assignable or appreciable quantity the fraction becomes greater than any assignable quantity. This is easily shewn by taking any fraction, as $\frac{a}{0{\cdot}1}$, which $= 10a$: for if the denominator be successively diminished one-tenth, we obtain the series of quantities $100a$, $1000a$, $10000a$, &c., proving that as the denominator of the fraction is diminished, the value of the fraction is increased, and without limit.

(6). Suppose that an expression involving x assumes the form $\frac{0}{0}$ when some particular value (a) is substituted for x, then it is clear that the expression is capable of being reduced to the form $\frac{p(x-a)^m}{q(x-a)^n}$, where p and q have no factor $x-a$ in them; and by dividing numerator and denominator by their highest common factor, the value of the fraction may be found when $x=a$. (See Art. 381.)

Thus it appears that a quantity which assumes the form $\frac{0}{0}$ may have a determinate value. And, conversely, since p is equal to $p\cdot\frac{x-a}{x-a}$, whatever be the values of x and a; if $x=a$, $p=\frac{0}{0}$, that is, any quantity may be made to assume the form $\frac{0}{0}$. But this is, in fact, multiplying and dividing by 0, on the supposition that the rules which apply to finite quantities apply also to 0 as a multiplier.

It may be said, generally, that to speak of absolute nothing as the subject of mathematical calculation is absurd; and that it can only become so when it is taken to represent some finite quantity in that state when by indefinite diminution it has become less than any appreciable quantity. The mathematical symbol 0 has, then, always reference to some other quantity from which it is derived; and it is this primitive quantity which must be the subject of our investigations when by becoming 0 it produces an anomalous result that requires to be explained.

That mistakes will constantly arise from considering 0 as an absolute quantity is easily seen: Thus, it has been shewn that $a^0=1=b^0$, therefore we might say, if 0 is a quantity, that $a=b$; or, since $2^0=4^0$, that $2=4$, both of which conclusions are manifestly wrong.

Again, if $x-1=0$, then

$$x(x-1)=x\times 0=0, \quad \text{also } x=1, \quad \text{and } \therefore\ x^2=1, \quad \text{or } x^2-1=0;$$

$$\therefore\ x(x-1)=0, \quad \text{and } (x+1)(x-1)=0,$$

$$\therefore\ x=x+1,$$

$$\text{or } 1=0, \text{ which is absurd.}$$

This amounts to saying that, because $a\times 0=0$, and $b\times 0=0$, therefore $a=b$, which is obviously incorrect.

461. *Given* $ax+b=cx+d$, *and* $\therefore\ x=\frac{d-b}{a-c}$, *to explain the result* (1) *when* $a=c$; *and* (2) *when* $a=c$, $b=d$.

(1) When $a=c$, $x=\frac{d-b}{0}=\infty$. In this case the proposed equation becomes $ax+b=ax+d$, which can only hold true on the supposition of x being such, that ax is not affected to any appreciable amount whichsoever of the two different quantities b or d be added to it, that is, x must be immeasurably great, agreeing with the result already found.

(2) When $a = c$, and $b = d$, $x = \frac{b-b}{a-a} = \frac{0}{0}$. In this case the original equation becomes $ax + b = ax + b$, an identity which is clearly satisfied by *any value whatever of* x; and this is the meaning of $\frac{0}{0}$ in this case.

462. *Given* $\left.\begin{aligned} \mathrm{ax} + \mathrm{by} &= \mathrm{c} \\ \textit{and}\ \mathrm{a'x} + \mathrm{b'y} &= \mathrm{c'} \end{aligned}\right\}$, *so that* $\mathrm{x} = \frac{\mathrm{b'c} - \mathrm{bc'}}{\mathrm{ab'} - \mathrm{a'b}}$, $\mathrm{y} = \frac{\mathrm{ac'} - \mathrm{a'c}}{\mathrm{ab'} - \mathrm{a'b}}$; *explain the results when* $\mathrm{a'} = \mathrm{ma}$, $\mathrm{b'} = \mathrm{mb}$, $\mathrm{c'} = \mathrm{mc}$.

Here $\frac{a'}{a} = m = \frac{b'}{b} = \frac{c'}{c}$, or $a'b = ab'$, $ac' = a'c$, $b'c = bc'$; $\therefore\ x = \frac{0}{0}$, and $y = \frac{0}{0}$. From the original equations we have $ax + by = c = \frac{c'}{m} = \frac{a'}{m}x + \frac{b'}{m}y = ax + by$, an identity which is satisfied by any values whatever of x and y; agreeing with the results before obtained.

463. *Given* $\mathrm{ax}^2 + \mathrm{bx} + \mathrm{c} = 0$, *and* $\therefore\ \mathrm{x} = \frac{-\mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}$; *explain this result when* $\mathrm{a} = 0$.

When $a = 0$, one value of x becomes $\frac{0}{0}$, the other $\frac{-2b}{0}$ or ∞. From the original equation, putting $\frac{1}{y}$ for x, we have

$$a + by + cy^2 = 0,$$

$$\text{and if } a = 0,\ by + cy^2 = 0, \text{ or } y(b + cy) = 0,$$

$$\therefore\ y = 0, \qquad \text{or } b + cy = 0,$$

$$\frac{1}{x} = 0, \qquad \text{or } b + \frac{c}{x} = 0,$$

$$x = \infty; \qquad \text{or } x = -\frac{c}{b}, \text{ which is the value}$$

of $\frac{0}{0}$ in this case, as may be easily shewn. Thus

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{4ac}{2a(-b - \sqrt{b^2 - 4ac})} = \frac{2c}{-b - \sqrt{b^2 - 4ac}} = \frac{-c}{b} \text{ when } a = 0.$$

464. *A* and *B* are travelling along the same road, and in the same direction, at a uniform pace of a miles and b miles per hour respectively. Given that at a known time *B* is d miles before *A*, find the time when they are together; and explain the result (1) when $a = b$, and (2) when $a < b$.

Let x be the number of hours from the known time to the time when they are together. Then in that time *A* travels ax miles, and *B* travels bx miles, and by the supposition

$$ax - d = bx,$$

$$\therefore\ x = \frac{d}{a - b}.$$

(1) Let $a = b$, then $x = \frac{d}{0} = \infty$; that is, A can never overtake B; which also evident from the circumstances; because, if A and B are once d iles apart and travel in the same direction *at the same pace,* they must ways be d miles apart, and can *never* come together.

(2) Let $a < b$, then x is negative, which signifies that the time of eir being together is past. For as A travels more slowly than B, it is ident they cannot at any *future* time come together, because the farther ey go the farther they are apart. But as by looking forward in time the stance between them keeps increasing, so by looking backward (supsing the journey continuous in that direction) that distance continually minishes, and $\frac{d}{b-a}$ hours *ago* it was 0, that is, A and B were then gether.

MAXIMA AND MINIMA. (From *Bourdon's* Alg.)

465. There is a class of problems which require for their solution determine the greatest or smallest values which an algebraical expresn will admit of by the variation of some quantity or quantities contained it. These problems are called Maxima and Minima Problems. Thus,

Prob. 1. Required to divide a given number $2a$ into two such parts at the product of the two parts may be the greatest possible.

Let $a + x$ be one part,

$\therefore\ a - x$ is the other.

The product is $a^2 - x^2$.

Let y be the greatest value required,

then $y = a^2 - x^2$,

and $x = \pm \sqrt{a^2 - y}$.

Now, that x may have a real value, y cannot be greater than a^2, but y be equal to a^2, which is therefore its greatest value. Hence, in that se, $x = 0$, and the two parts of $2a$ required are equal to each other.

Prob. 2. Required the minimum of $\frac{a^2x^2+b^2}{(a^2-b^2)x}$, when $a > b$.

$$\text{Let } \frac{a^2x^2+b^2}{(a^2-b^2)x} = y, \quad \text{then}$$

$$a^2x^2 + b^2 = (a^2-b^2)yx,$$

$$a^2x^2 - (a^2-b^2)yx = -b^2,$$

$$x^2 - \frac{(a^2-b^2)}{a^2}yx + \frac{(a^2-b^2)^2}{4a^4}y^2 = \frac{(a^2-b^2)^2}{4a^4}y^2 - \frac{b^2}{a^2};$$

$$\therefore\ x = \frac{a^2-b^2}{2a^2}\cdot y \pm \frac{1}{2a^2}\sqrt{(a^2-b^2)^2y^2 - 4a^2b^2}.$$

Now, that x may have a real value, $(a^2-b^2)^2y^2$ must not be less than $4a^2b^2$, but it may be equal to it, or $y=\frac{2ab}{a^2-b^2}$, which is therefore its *minimum*, or least value.

Hence also in this case, $x=\frac{a^2-b^2}{2a^2}\cdot\frac{2ab}{a^2-b^2}=\frac{b}{a}$.

466. If the quantity under the radical sign remains positive whatever value be given to y, then the proposed quantity will admit of neither a *maximum* nor a *minimum*.

Ex. Required to determine whether $\frac{4x^2+4x-3}{6(2x+1)}$ admits of either a *maximum* or a *minimum*.

$$\text{Let } \frac{4x^2+4x-3}{6(2x+1)}=y, \text{ then}$$

$$4x^2-4(3y-1)x=6y+3,$$

$$\text{and } x=\frac{3y-1}{2}\pm\tfrac{1}{2}\sqrt{9y^2+4}.$$

Now, whatever value may be given to y, the quantity under the root will be always positive; therefore the proposed expression does not admit of any *maximum* or *minimum*.

467. If the quantity under the root be of the form my^2+ny+p, then by solving the equation $my^2+ny+p=0$, we can find the greatest or least value of y which will permit $\sqrt{my^2+ny+p}$ to be real, and therefore the required *maximum* or *minimum*.

Ex. 1. Let a and b be two numbers of which $a>b$; required the greatest value of $\frac{(x+a)(x-b)}{x^2}$. Ans. *Maximum* $=\frac{(a+b)^2}{4ab}$; and $x=\frac{2ab}{a-b}$.

Ex. 2. Required the smallest value of $\frac{(a+x)(b+x)}{x}$.

Ans. *Minimum* $=(\sqrt{a}+\sqrt{b})^2$; and $x=\sqrt{ab}$.

APPLICATION OF ALGEBRA TO GEOMETRY.

468. The signs made use of in algebraical calculations being general, the conclusions obtained by their assistance may, with great ease and convenience, be transferred from abstract magnitudes to every class of particular quantities; thus, the relations of lines, surfaces, or solids, may generally be deduced from the principles of

Algebra, and many properties of these quantities discovered, which could not have been derived from principles purely geometrical.

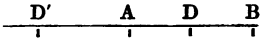
469. *Simple algebraical quantities may be represented by lines.*

Any line, AB, may be taken at pleasure to represent one quantity a; but if we have a second quantity, b, to represent, we must take a line which has to the former line the same ratio that b has to a.

Instead of saying AB represents a, we may say $AB = a$, supposing AB to contain as many linear units as a contains numeral ones.

470. *When a series of algebraical quantities is to be represented on one line, and each of them measured from the same point, the positive quantities being represented by lines taken in one direction, the negative quantities must be represented by lines taken in the opposite direction.*

Let a be the greatest of these quantities, then $a - x$ may, by the variation of x, become equal to each of them in succession. Let AB be the given line, and A the point from which the quantities are to be measured; take AB equal to a; and since $a - x$ must be measured from A,



BD must be taken in the contrary direction equal to x, then $AD = a - x$; and that $a - x$ may successively coincide with each quantity in the series, beginning with the greatest positive quantity, x must increase; therefore BD, which is equal to x, must increase; and when x is greater than a, BD is greater than AB, and AD', which represents the negative quantity $a - x$, lies in the opposite direction from A.

Cor. 1. If the algebraical value of a line be found to be negative, the line must be measured in a direction opposite to that which, in the investigation, we supposed to be positive.

Cor. 2. If quantities be measured upon a line from its intersection with another, the positive quantities being taken in one direction, the negative quantities must be taken in the other.

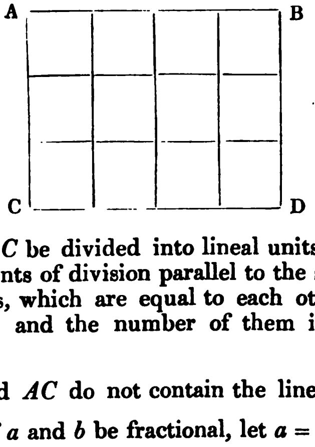
471. If a fourth proportional to lines representing p, q, r be taken, it will represent $\frac{qr}{p}$; and if $p = 1$, it will represent qr; if also q and r be equal, it will represent q^2.

472. If a mean proportional between lines representing a and b be taken, it will represent $\sqrt{ab}$, which, when $a = 1$, becomes $\sqrt{b}$.

Hence it appears that any possible algebraical quantities may be represented by lines; and conversely, lines may be expressed algebraically; and if the relations of the algebraical quantities be known, the relations of the lines are known.

473. *The area included within a rectangular parallelogram may be measured by the product of the two numbers which measure two adjacent sides.*

Let the sides AB, AC of the rectangular parallelogram AD be measured by the lineal quantities a, b, respectively; then $a \times b$ will express the number of superficial units in the area, that is, the number of squares it contains, each of which is described upon a lineal unit. For instance, if the lineal unit be a foot, of which AB contains a, and AC contains b, the parallelogram AD contains $a \times b$ square feet.



For, 1st, if AB, AC be divided into lineal units, and straight lines be drawn through the points of division parallel to the sides, the whole figure is made up of squares, which are equal to each other, and to the square upon the lineal unit; and the number of them is evidently a taken b times, or $a \times b$.

2ndly. If AB and AC do not contain the lineal unit an exact number of times, that is, if a and b be fractional, let $a = \alpha + \frac{1}{m}$, and $b = \beta + \frac{1}{n}$. Then take another lineal unit which is $\frac{1}{mn}$th part of the former; and by what has been shewn the square described upon the larger unit contains $mn \times mn$ of that described upon the smaller. Again, the sides AB, AC respectively contain $mn\alpha + n$, $mn\beta + m$ lineal units of the smaller kind, and therefore, by the first case, the whole figure contains $(mna+n)\times(mn\beta+m)$ square units of the smaller kind; that is, the area

$$=m^2n^2\left\{\alpha+\frac{1}{m}\right\}.\left\{\beta+\frac{1}{n}\right\} \text{ of the smaller units,}$$

$$=\left(\alpha+\frac{1}{m}\right)\left(\beta+\frac{1}{n}\right) \text{ of the larger units,}$$

$$=ab, \text{ as before.}$$

3dly. If the sides AB, AC be *incommensurable* with the lineal unit, a unit may be found which is commensurable with certain lines that approach *as near as we please* to AB, and AC, and therefore the product of such lines will represent the area of a rectangle differing from the rectangle AD by a quantity less than any that can be assigned, that is, we may, in this case also, without error express AD by $AB\times AC$. (See Art. 260.)

474. COR. 1. Since, by *Euclid*, Book I. Prop. 35, the area of an oblique-angled parallelogram is equal to that of the rectangular parallelogram upon the same base and between the same parallels, therefore

area of *any* parallelogram = base × altitude.

475. COR. 2. Also, since by *Euclid*, Book I. Prop. 41, the area of a triangle is half that of the parallelogram upon the same base and between the same parallels, therefore

$$\text{area of } any \text{ triangle} = \frac{1}{2}\times\text{base}\times\text{altitude}.$$

476. COR. 3. Since any rectilineal figure may be divided into triangles, its area may be found by taking the sum of all the triangles.

477. *The solid content or volume included within a rectangular parallelopiped may be measured by the product of the three numbers which measure its length, breadth, and height.*

Let the base of the parallelopiped be divided into its component squares, as in the preceding Proposition, and through each of the parallels suppose planes drawn at right-angles to the base; and let the same thing be done with one of the faces adjacent to the base. Then it is evident that the whole figure is divided into a certain number of equal cubes, each cube having for its face one of the squares described upon the lineal unit; (that is, if the lineal unit be a foot, each of these cubes will have its length, breadth, and height equal to a foot, and is called a cubic foot). Now the *number* of these cubes is manifestly equal to the number of squares in the base taken as many times as there are lineal units in the height; therefore

$$\begin{aligned}\text{content or volume} &= \text{base}\times\text{height}\\ &= \text{length}\times\text{breadth}\times\text{height}.\end{aligned}$$

COR. *Any* three of these quantities being given, the fourth may be found. Thus, if C be the content, l the length, b the breadth, and h the height, we have

$$l=\frac{C}{bh},\quad b=\frac{C}{hl},\quad h=\frac{C}{bl}.$$

The following Examples will sufficiently illustrate the preceding Theory, for our present purpose:—

Ex. 1. *If a straight line be divided into any two parts, the squares of the whole line, and of one of the parts, are equal to twice the rectangle contained by the whole and that part, together with the square of the other part.* (Euclid, Book II. Prop. 7.)

Let a and b represent the two parts into which the given line is divided; then $a+b$ is the whole line,

and $(a+b)^2+a^2$ = the squares of the whole line, and of one of the parts;

$$=a^2+2ab+b^2+a^2$$
$$=2a^2+2ab+b^2$$
$$=2(a+b)a+b^2$$

= twice the rectangle &c. together with the square of the other part. Q. E. D.

Ex. 2. *To find the radius of a circle inscribed in a given triangle.*

See Euclid's diagram, Book III. Prop. 4; let r be the radius of the inscribed circle, and a, b, c the sides of the triangle respectively opposite to the angles A, B, C. Then (Art. 475)

$$\frac{1}{2}r\,.\,a+\frac{1}{2}r\,.\,b+\frac{1}{2}r\,.\,c=\text{whole area of the triangle,}$$

or $\frac{1}{2}r(a+b+c)=\frac{1}{2}a\times$ the perpend^r (p) upon a from the opposite angle;

$$\text{and } \therefore\ r=\frac{a}{a+b+c}\,.\,p.$$

To find p, let the segments into which a is divided by it be x and $a-x$; then (Euclid, Book I. Prop. 47)

$$c^2-x^2=p^2=b^2-(a-x)^2$$
$$c^2=b^2-a^2+2ax;$$
$$\therefore\ x=\frac{a^2+c^2-b^2}{2a},$$

$$\text{and } p^2=c^2-\left(\frac{a^2+c^2-b}{2a}\right)^2=\frac{(2ac)^2-(a^2+c^2-b^2)^2}{4a^2}$$

$$=\frac{(a+b+c)(a+c-b)(a+b-c)(b+c-a)}{4a^2};$$

$$\therefore\ p=\frac{1}{2a}\sqrt{(a+b+c)(a+c-b)(a+b-c)(b+c-a)},$$

$$\text{and } r=\frac{1}{2}\sqrt{\frac{(a+c-b)(a+b-c)(b+c-a)}{a+b+c}}.$$

Ex. 3. *To find the area of the square inscribed in a given circle; and o of the square circumscribed about a given circle.*

(1). Let r be the radius of the circle; then (see EUCLID, Book IV. op. 6.)

$$r^2+r^2=AD^2,$$

$$\text{or }\ 2r^2=\text{inscribed square.}$$

(2). Again, $r^2+r^2+r^2+r^2$ or $4r^2$ = circumscribed square,

= twice the inscribed square.

Ex. 4. *To find the area of the equilateral and equiangular hexagon scribed in a given circle.*

Let r be the radius of the given circle, then (EUCLID, Book IV. op. 15.) also r = the side of the inscribed hexagon; and the area the hexagon = the sum of the areas of the six equal equilateral triangles which it is composed,

$$=6\times\frac{1}{2}r\,.\sqrt{r^2-\frac{r^2}{4}}=\frac{3\sqrt{3}}{2}\,.\,r^2.$$

Ex. 5. *The depth of water in a cistern (whose form is a rectangular rallelopiped) is* h *feet, and the base contains* a *square feet. Find (1) the mber of cubic feet of water; and (2) the depth of the same quantity water in another cistern whose base contains* b *square inches.*

(1). The quantity of water = base × depth = ah cubic feet.

(2). The depth for 2nd cistern $=\dfrac{\text{quantity of water (in cubic feet)}}{\text{base (in square feet)}}$

$$=ah\div\frac{b}{144}=\frac{144ah}{b}\text{ feet.}$$

It is not necessary here to multiply Examples, because the subject is ow of sufficient importance to form a separate treatise, to which, in the gular course of reading, the student's attention will be next directed. r. *Hymers' Treatise on Conic Sections and the Application of Algebra to eometry* is the book in most general use for teaching Algebraic eometry.

APPENDIX.

EQUATIONS.

In the infinite variety of equations which ingenious persons may put gether, it is not to be supposed that any general Rules can be laid down every operation necessary to their solution. There are, however, culiar artifices of more frequent occurrence than others, which shall be hibited in the following Examples.

Ex. 1. $\frac{4x-17}{x-4}+\frac{10x-13}{2x-3}=\frac{8x-30}{2x-7}+\frac{5x-4}{x-1}$; find x.

$$\frac{4(x-4)-1}{x-4}+\frac{5(2x-3)+2}{2x-3}=\frac{4(2x-7)-2}{2x-7}+\frac{5(x-1)+1}{x-1},$$

$$4-\frac{1}{x-4}+5+\frac{2}{2x-3}=4-\frac{2}{2x-7}+5+\frac{1}{x-1},$$

$$\frac{2}{2x-3}-\frac{1}{x-4}=\frac{1}{x-1}-\frac{2}{2x-7},$$

$$\frac{-5}{2x^2-11x+12}=\frac{-5}{2x^2-9x+7},$$

$$2x^2-9x+7=2x^2-11x+12,$$

$$2x=5,$$

$$x=\frac{5}{2}=2\frac{1}{2}.$$

Ex. 2. $\frac{a+x+\sqrt{2ax+x^2}}{a+x-\sqrt{2ax+x^2}}=b^2$; find x.

By Art. 195. $\frac{a+x}{\sqrt{2ax+x^2}}=\frac{b^2+1}{b^2-1},$

$$\frac{a^2+2ax+x^2}{2ax+x^2}=\frac{a^2}{2ax+x^2}+1=\left(\frac{b^2+1}{b^2-1}\right)^2,$$

$$\frac{a^2}{2ax+x^2}=\left(\frac{b^2+1}{b^2-1}\right)^2-1=\frac{4b^2}{(b^2-1)^2},$$

$$\frac{2ax+x^2}{a^2}=\frac{(b^2-1)^2}{4b^2},$$

$$\left(\frac{a+x}{a}\right)^2 = \frac{(b^2-1)^2}{4b^2} + 1 = \frac{(b^2+1)^2}{4b^2},$$

$$\frac{a+x}{a} = \frac{b^2+1}{2b},$$

$$\frac{x}{a} = \frac{b^2+1-2b}{2b} = \frac{(b-1)^2}{2b},$$

$$\therefore\ x = \frac{a}{2b}(b-1)^2.$$

Ex. 3. $\dfrac{a-x+\sqrt{2ax-x^2}}{a-x} = b$; find x.

$$1 + \frac{\sqrt{2ax-x^2}}{a-x} = b,$$

$$\frac{2ax-x^2}{a^2-2ax+x^2} = (b-1)^2,$$

$$\left(\frac{a}{a-x}\right)^2 - 1 = (b-1)^2,$$

$$\left(\frac{a}{a-x}\right)^2 = 1 + (b-1)^2,$$

$$\frac{a}{a-x} = \sqrt{1+(b\quad 1)^2};$$

$$\frac{a-x}{a} = \frac{1}{\sqrt{1+(b-1)^2}},$$

$$\frac{x}{a} = 1 - \frac{1}{\sqrt{1+(b-1)^2}},$$

$$\therefore\ x = a - \frac{a}{\sqrt{1+(b-1)^2}}.$$

Ex. 4. $\dfrac{243+324\sqrt{3x}}{16x-3} = (4\sqrt{x}-\sqrt{3})^2$; find x.

$$\frac{81\sqrt{3}\,.\,(\sqrt{3}+4\sqrt{x})}{(4\sqrt{x}+\sqrt{3})\,(4\sqrt{x}-\sqrt{3})} = (4\sqrt{x}-\sqrt{3})^2,$$

$$\frac{81\sqrt{3}}{4\sqrt{x}-\sqrt{3}} = (4\sqrt{x}-\sqrt{3})^2,$$

$$(4\sqrt{x}-\sqrt{3})^3 = 81\sqrt{3},$$

$$4\sqrt{x}-\sqrt{3} = 3\sqrt{3},$$

$$4\sqrt{x} = 4\sqrt{3},$$

$$\therefore\ x = 3.$$

Ex. 5. $\sqrt[3]{a+x}+\sqrt[3]{a-x}=b$; find x.

$$(\sqrt[3]{a+x}+\sqrt[3]{a-x})^3=2a+3\sqrt[3]{a^2-x^2}\,.\,(\sqrt[3]{a+x}+\sqrt[3]{a-x}),$$

and $\sqrt[3]{a+x}+\sqrt[3]{a-x}$ by the supposition is equal to b;

∴ cubing both sides of the proposed equation,

$$2a+3\sqrt[3]{a^2-x^2}\,.\,b=b^3,$$

$$3b\sqrt[3]{a^2-x^2}=b^3-2a,$$

$$\sqrt{a^2-x^2}=\frac{b^2}{3}-\frac{2a}{3b},$$

$$a^2-x^2=\left(\frac{b^2}{3}-\frac{2a}{3b}\right)^3,$$

$$x^2=a^2-\left(\frac{b^2}{3}-\frac{2a}{3b}\right)^3,$$

$$\therefore\ x=\sqrt{a^2-\left(\frac{b^2}{3}-\frac{2a}{3b}\right)^3}.$$

Ex. 6. $x-1=2+\frac{2}{\sqrt{x}}$; find x.

$$x-1=\frac{2}{\sqrt{x}}(\sqrt{x}+1),$$

∴ dividing by $\sqrt{x}+1$, $\sqrt{x}-1=\frac{2}{\sqrt{x}}$,

$\therefore\ \sqrt{x}+1=0$, or $\sqrt{x}=-1$, and $\therefore\ x=1$. (Art. 208.)

Also $x-\sqrt{x}=2$,

$$x-\sqrt{x}+\frac{1}{4}=2+\frac{1}{4}=\frac{9}{4},$$

$$\sqrt{x}-\frac{1}{2}=\pm\frac{3}{2},$$

$$\sqrt{x}=\frac{1\pm 3}{2}=2,\ \text{or}\ -1,$$

$$\therefore\ x=4,\ \text{or}\ 1.$$

Ex. 7. $x^3-3x=2$; find x.

$$x^3-x=2x+2,$$

$$\therefore\ x(x^2-1)=2(x+1),$$

$\therefore\ x+1=0$, or $x=-1$. (Art. 208).

Also $x(x-1)=2$, or $x^2-x=2$,

$$x^2 - x + \frac{1}{4} = \frac{9}{4},$$

$$x - \frac{1}{2} = \pm \frac{3}{2},$$

$$\therefore \quad x = \frac{1 \pm 3}{2} = 2, \text{ or } -1.$$

Ex. 8. $x^2 - \dfrac{2}{3x} = 1\frac{4}{9}$; find x.

$$x^2 - \frac{4}{9} = 1 + \frac{2}{3x},$$

$$\left(x + \frac{2}{3}\right)\left(x - \frac{2}{3}\right) = \frac{1}{x}\left(x + \frac{2}{3}\right),$$

$$\therefore \quad x + \frac{2}{3} = 0, \quad \text{or } x = -\frac{2}{3} \ldots\ldots\ldots\ldots(\text{I}).$$

Also $x - \dfrac{2}{3} = \dfrac{1}{x}$,

$$x^2 - \frac{2}{3}x = 1,$$

$$x^2 - \frac{2}{3}x + \left(\frac{1}{3}\right)^2 = 1 + \frac{1}{9} = \frac{10}{9},$$

$$x - \frac{1}{3} = \pm \frac{\sqrt{10}}{3},$$

$$\therefore \quad x = \frac{1 \pm \sqrt{10}}{3} \ldots\ldots\ldots\ldots\ldots(\text{II}).$$

Ex. 9. $\sqrt[m]{(1+x)^2} - \sqrt[m]{(1-x)^2} = \sqrt[m]{1-x^2}$; find x.

Converting the roots into fractional indices, the equation is

$$(1+x)^{\frac{2}{m}} - (1-x)^{\frac{2}{m}} = (1-x^2)^{\frac{1}{m}} = (1+x)^{\frac{1}{m}}(1-x)^{\frac{1}{m}}.$$

Dividing by $(1-x)^{\frac{2}{m}}$, $\left(\dfrac{1+x}{1-x}\right)^{\frac{2}{m}} - 1 = \left(\dfrac{1+x}{1-x}\right)^{\frac{1}{m}}$,

$$\therefore \quad \left(\frac{1+x}{1-x}\right)^{\frac{2}{m}} - \left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} + \frac{1}{4} = 1 + \frac{1}{4} = \frac{5}{4},$$

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} - \frac{1}{2} = \pm \frac{\sqrt{5}}{2},$$

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{m}} = \frac{1 \pm \sqrt{5}}{2},$$

$$\frac{1+x}{1-x} = \frac{(1 \pm \sqrt{5})^m}{2^m},$$

$$\therefore \; x = \frac{(1 \pm \sqrt{5})^m - 2^m}{(1 \pm \sqrt{5})^m + 2^m}. \quad \text{(Art. 195).}$$

Ex. 10. $\dfrac{1+x^4}{(1+x)^4} = a$; find x.

$$1 + x^4 = a(1+x)^4$$

$$= a(1 + 4x + 6x^2 + 4x^3 + x^4),$$

$$\therefore \; (1-a)(1+x^4) = 4a(x + x^3) + 6ax^2;$$

dividing by x^2, $(1-a)\left(x^2 + \dfrac{1}{x^2}\right) = 4a\left(x + \dfrac{1}{x}\right) + 6a$,

$$\therefore \; x^2 + \frac{1}{x^2} - \frac{4a}{1-a}\left(x + \frac{1}{x}\right) = \frac{6a}{1-a},$$

$$\left(x + \frac{1}{x}\right)^2 - \frac{4a}{1-a}\left(x + \frac{1}{x}\right) = \frac{6a}{1-a} + 2 = \frac{2+4a}{1-a},$$

$$+ \left.\frac{1}{x}\right)^2 - \frac{4a}{1-a}\left(x + \frac{1}{x}\right) + \left(\frac{2a}{1-a}\right)^2 = \frac{2+4a}{1-a} + \frac{4a^2}{(1-a)^2} = \frac{2(1+a)}{(1-a)^2},$$

$$\therefore \; x + \frac{1}{x} - \frac{2a}{1-a} = \pm \frac{\sqrt{2(1+a)}}{1-a},$$

$$x + \frac{1}{x} = \frac{2a \pm \sqrt{2(1+a)}}{1-a} = 2p, \text{ suppose};$$

$$\therefore \; x^2 - 2px = -1,$$

$$x^2 - 2px + p^2 = p^2 - 1,$$

$$x - p = \pm \sqrt{p^2 - 1},$$

$$\therefore \; x = p \pm \sqrt{p^2 - 1}.$$

Ex. 11. $x + a + 3\sqrt[3]{abx} = b$; find x.

Assume $\sqrt[3]{x} + \sqrt[3]{a} = \sqrt[3]{y}$,

$\therefore$ cubing these equals $x + a + 3\sqrt[3]{ax}\,.\,(\sqrt[3]{x} + \sqrt[3]{a}) = y$;

$$\therefore \; x + a + 3\sqrt[3]{ayx} = y.$$

But, comparing this with the original equation, it appears that $y =$

$$\therefore\ \sqrt[3]{x} + \sqrt[3]{a} = \sqrt[3]{b},$$

$$\sqrt[3]{x} = \sqrt[3]{b} - \sqrt[3]{a},$$

$$\therefore\ x = (\sqrt[3]{b} - \sqrt[3]{a})^3.$$

Ex. 12. $x^3 - 1 = 0$; find all the values of x.

$$x^3 - 1 = (x - 1)(x^2 + x + 1) = 0,$$

$$\therefore\ x - 1 = 0, \text{ or } x = 1 \ldots\ldots\ldots\ldots \text{(I)}.$$

Also $x^2 + x + 1 = 0$,

$$x^2 + x + \frac{1}{4} = \frac{1}{4} - 1 = -\frac{3}{4}$$

$$x + \frac{1}{2} = \pm\frac{\sqrt{-3}}{2},$$

$$\therefore\ x = \frac{-1 \pm \sqrt{-3}}{2} \ldots\ldots\ldots\ldots \text{(II)}.$$

Ex. 13. $x^4 + 1 = 0$; find all the values of x.

Dividing by x^2, $x^2 + \frac{1}{x^2} = 0$,

$$x^2 + 2 + \frac{1}{x^2} = 2,$$

$$x + \frac{1}{x} = \pm\sqrt{2},$$

$$x^2 \mp \sqrt{2}\,.\,x = -1,$$

$$x^2 \mp \sqrt{2}\,.\,x + \frac{1}{2} = \frac{1}{2} - 1 = -\frac{1}{2},$$

$$x \mp \frac{1}{\sqrt{2}} = \frac{\pm\sqrt{-1}}{\sqrt{2}},$$

$$\therefore\ x = \frac{\pm 1 \pm \sqrt{-1}}{\sqrt{2}}.$$

In a similar manner to that of the last two Examples may the roo of $x^5 - 1 = 0$, $x^5 + 1 = 0$, $x^6 + 1 = 0$, and like equations, be easily found.

Ex. 14. $(x - a)\sqrt{x} - (x + a)\sqrt{b} = b(\sqrt{x} - \sqrt{b})$; find x.

$$x\sqrt{x} - a\sqrt{x} - x\sqrt{b} - a\sqrt{b} = b(\sqrt{x} - \sqrt{b}),$$

$$a(\sqrt{x} + \sqrt{b}) = (x - b)(\sqrt{x} - \sqrt{b}) = (\sqrt{x} + \sqrt{b})(\sqrt{x} - \sqrt{b})^2,$$

$$\therefore\ \sqrt{x}+\sqrt{b}=0,\ \sqrt{x}=-\sqrt{b},\ \text{and}\ x=b\ \ldots\ldots\ldots\ldots\ldots\ (\text{I});$$

Also $\sqrt{x}-\sqrt{b}=\sqrt{a},$

$$\sqrt{x}=\sqrt{a}+\sqrt{b},\ \text{or}\ x=(\sqrt{a}+\sqrt{b})^2\ \ldots\ldots\ldots\ldots\ (\text{II}).$$

Ex. 15. $a+x+\sqrt{2ax+x^2}=\sqrt{ax-x^2}+\sqrt{2a^2-ax-x^2}$; find x.

$$a+x-\sqrt{ax-x^2}=\sqrt{2a^2-ax-x^2}-\sqrt{2ax+x^2}$$

$$=\sqrt{2a+x}.\{\sqrt{a-x}-\sqrt{x}\};$$

$$a^2+3ax-2(a+x)\sqrt{ax-x^2}=(2a+x)\{a-2\sqrt{ax-x^2}\},$$

$$2a\sqrt{ax-x^2}=a^2-2ax,$$

$$2\sqrt{ax-x^2}=a-2x,$$

$$4ax-4x^2=a^2-4ax+4x^2,$$

$$8x^2-8ax=-a^2,$$

$$x^2-ax=-\frac{a^2}{8},$$

$$x^2-ax+\frac{a^2}{4}=\frac{2a^2}{16},$$

$$x-\frac{a}{2}=\pm\frac{a}{4}\sqrt{2},$$

$$\therefore\ x=\frac{a}{2}\{1\pm\frac{1}{2}\sqrt{2}\}.$$

Ex. 16. $a+(b+\sqrt{x})\sqrt{x}=(b-\sqrt{x})\sqrt{2a+x}$; find x.

$$a+x+\sqrt{2ax+x^2}=b(\sqrt{2a+x}-\sqrt{x}),$$

$$(a+x+\sqrt{2ax+x^2})^2=b^2(2a+2x-2\sqrt{2ax+x^2})$$

$$=2b^2(a+x-\sqrt{2ax+x^2}),$$

$$(a+x+\sqrt{2ax+x^2})^3=2b^2.\{\overline{a+x}|^2-\overline{2ax+x^2}\}$$

$$=2a^2b^2,$$

$$a+x+\sqrt{2ax+x^2}=\sqrt[3]{2a^2b^2},$$

$$2ax+x^2=ab\sqrt[3]{4ab}-2\sqrt[3]{2a^2b^2}(a+x)+a^2+2ax+x^2,$$

$$a+x=\frac{a^2}{2\sqrt[3]{2a^2b^2}}+\frac{ab}{2}\sqrt[3]{\frac{2}{ab}},$$

$$\therefore\ x=\frac{a}{2}\sqrt[3]{\frac{a}{2b^2}}+\frac{ab}{2}\sqrt[3]{\frac{2}{ab}}-a.$$

Ex. 17. $\frac{2a\sqrt{1+x^2}}{1-x+\sqrt{1+x^2}} = a+b\,1$; find x.

$$2a\sqrt{1+x^2} = (a+b)(1-x) + (a+b)\sqrt{1+x^2},$$

$$(a-b)\sqrt{1+x^2} = (a+b)(1-x),$$

$$\frac{\sqrt{1+x^2}}{1-x} = \frac{a+b}{a-b},$$

$$\frac{1+x^2-2x}{1+x^2} = \left(\frac{a-b}{a+b}\right)^2,$$

$$\frac{2x}{1+x^2} = 1 - \frac{(a-b)^2}{(a+b)^2} = \frac{4ab}{(a+b)^2},$$

$$\frac{(1+x)^2}{(1-x)^2} = \frac{(a+b)^2+4ab}{(a+b)^2-4ab} = \frac{(a-b)^2+8ab}{(a-b)^2},$$

$$\therefore \frac{1+x}{1-x} = \sqrt{1+\frac{8ab}{(a-b)^2}};$$

$$\therefore x = \frac{\sqrt{1+\frac{8ab}{(a-b)^2}}-1}{\sqrt{1+\frac{8ab}{(a-b)^2}}+1} = \frac{(a-b)^2}{8ab}.\left\{\sqrt{1+\frac{8ab}{(a-b)^2}}-1\right\}^2.$$

Ex. 18. $\frac{x-\sqrt{x^2-a^2}}{\sqrt{x+\sqrt{x^2-a^2}}} = \sqrt[4]{x^2-a^2}.\{\sqrt{x^2+ax}-\sqrt{x^2-ax}\}$; find x.

Squaring, $\frac{(x-\sqrt{x^2-a^2})^2}{x+\sqrt{x^2-a^2}} = \sqrt{x^2-a^2}.2x\{x-\sqrt{x^2-a^2}\}$,

$$\therefore \frac{x-\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}} = 2x\sqrt{x^2-a^2},$$

$$\frac{(x-\sqrt{x^2-a^2})^2}{a^2} = 2x\sqrt{x^2-a^2},$$

$$2x^2-a^2-2x\sqrt{x^2-a^2} = 2a^2x\sqrt{x^2-a^2},$$

$$2x^2-a^2 = 2x(a^2+1)\sqrt{x^2-a^2},$$

$$4x^4-4a^2x^2+a^4 = 4a^4x^4+8a^2x^4+4x^4-4a^6x^2-8a^4x^2-4a^2x^2,$$

$$a^4 = 4a^4x^4+8a^2x^4-4a^6x^2-8a^4x^2,$$

$$a^2 = 4(a^2+2)x^4-4a^2(a^2+2)x^2,$$

$$x^4 - a^2x^2 = \frac{a^2}{4(a^2+2)}$$

$$x^4 - a^2x^2 + \frac{a^4}{4} = \frac{a^2}{4} \cdot \frac{a^4 + 2a^2 + 1}{a^2+2},$$

$$x^2 = \frac{a}{2}\left\{a \pm \frac{a^2+1}{\sqrt{a^2+2}}\right\},$$

$$\therefore\ x = \pm\sqrt{\frac{a}{2}}\left\{a \pm \frac{a^2+1}{\sqrt{a^2+2}}\right\}^{\frac{1}{2}}.$$

x. 19. $x - 2\sqrt{x+2} = 1 + \sqrt[4]{x^3 - 3x + 2}$; find x.

$$x - 1 - 2\sqrt{x+2} = (x^2 - 2x + 1)^{\frac{1}{4}} . (x+2)^{\frac{1}{4}}$$

$$= \sqrt{x-1} . \sqrt[4]{x+2}$$

$$x - 1 - \sqrt{x+2} = \sqrt[4]{x+2} . \{\sqrt{x-1} + \sqrt[4]{x+2}\};$$

$\therefore \sqrt{x-1} + \sqrt[4]{x+2} = 0$; and $\sqrt{x-1} - \sqrt[4]{x+2} = \sqrt[4]{x+2}$;

or $\sqrt{x-1} = -\sqrt[4]{x+2}$; and $\sqrt{x-1} = 2\sqrt[4]{x+2}$,

$$x^2 - 2x + 1 = x + 2;$$

$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = \frac{13}{4};$$

$$x - \frac{3}{2} = \frac{\pm\sqrt{13}}{2};$$

$$\therefore\ x = \frac{3 \pm \sqrt{13}}{2} \ldots \text{(I)}.$$

$$x^2 - 2x + 1 = 16x + 32,$$

$$x^2 - 18x + 81 = 112,$$

$$x - 9 = \pm 4\sqrt{7},$$

$$x = 9 \pm 4\sqrt{7} \ldots \text{(II)}.$$

20. $\dfrac{2x^2 + 1 + x\sqrt{4x^2+3}}{2x^2 + 3 + x\sqrt{4x^2+3}} = a$; find x.

$$1 - \frac{2x^2 + 1 + x\sqrt{4x^2+3}}{2x^2 + 3 + x\sqrt{4x^2+3}} = \frac{2}{2x^2 + 3 + x\sqrt{4x^2+3}} = 1 - a,$$

$$2x^2 + 3 + x\sqrt{4x^2+3} = \frac{2}{1-a},$$

$$x\sqrt{4x^2+3} = \frac{2}{1-a} - 3 - 2x^2 = \frac{3a-1}{1-a} - 2x^2,$$

$$4x^4 + 3x^2 = \left(\frac{3a-1}{1-a}\right)^2 - 4 . \frac{3a-1}{1-a}\, x^2 + 4x^4,$$

$$\left(\frac{12a-4}{1-a} + 3\right)x^2 = \left(\frac{3a-1}{1-a}\right)^2,$$

$$(9a-1)x^2=\frac{(3a-1)^2}{1-a};$$

$$\therefore\ x=\pm\frac{3a-1}{\sqrt{(1-a)(9a-1)}}.$$

Ex. 21. $\dfrac{(a^2-1)a+a^2x-x\sqrt{2a^2-1}}{(a^2-1)a+a^2x+x\sqrt{2a^2-1}}=(1-a^2)(a+x)^2-2ax$; find x.

Multiplying the numerator and denominator of the fraction by the denominator, the numerator becomes

$$\{(a^2-1)a+a^2x\}^2-x^2(2a^2-1),$$

$$\text{or }\ (a^2-1)^2a^2+2a^3x(a^2-1)+a^4x^2-2a^2x^2+x^2,$$

$$\text{or }\ (a^2-1)^2a^2+2a^3x(a^2-1)+x^2(a^2-1)^2,$$

$$\text{or }\ (a^2-1)\{a^2(a^2-1)+2a^3x+x^2(a^2-1)\},$$

$$\text{or }\ (a^2-1)\{a^4+2a^3x+a^2x^2-a^2-x^2\},$$

$$\text{or }\ (a^2-1)\{a^2(a+x)^2-(a^2+x^2)\}.$$

Also $(1-a^2)(a+x)^2-2ax=(a+x)^2-2ax-a^2(a+x)^2,$

$$=a^2+x^2-a^2(a+x)^2,$$

$$=-\{a^2(a+x)^2-(a^2+x^2)\}.$$

Hence $\dfrac{a^2-1}{(\text{denom}^{\text{r}}.)^2}=-1;$

$$\therefore\ \text{denom}^{\text{r}}.=\pm\sqrt{1-a^2},$$

$$\text{or }\ (a^2-1)a+a^2x+x\sqrt{2a^2-1}=\pm\sqrt{1-a^2};$$

$$\therefore\ (a^2+\sqrt{2a^2-1})x=\pm\sqrt{1-a^2}.\{1\pm a\sqrt{1-a^2}\};$$

$$\therefore\ x=\pm\sqrt{1-a^2}\cdot\frac{1\pm a\sqrt{1-a^2}}{a^2+\sqrt{2a^2-1}}.$$

Ex. 22. $\dfrac{1-ax+\sqrt{1+a^2}-a\sqrt{1+x^2}}{1-ax+\sqrt{1+x^2}-x\sqrt{1+a^2}}=a$; find x.

$$1-ax+\sqrt{1+a^2}-a\sqrt{1+x^2}=a(1-ax)+a\sqrt{1+x^2}-ax\sqrt{1+a^2},$$

$$(ax-1)(a-1)+(ax+1)\sqrt{1+a^2}=2a\sqrt{1+x^2};$$

$$\therefore\ (a^2x^2-2ax+1)(a-1)^2+2(a-1)\sqrt{1+a^2}.(a^2x^2-1)$$
$$+(a^2x^2+2ax+1)(1+a^2)=4a^2+4a^2x^2,$$

$$2(a^2x^2+1)(1+a^2)-2a(ax-1)^2+2(a-1)\sqrt{1+a^2}.(a^2x^2-1)=4a^2+4a^2x^2,$$

$$(a^2x^2+1)(1+a^2)-a(ax-1)^2+(a-1)\sqrt{1+a^2}(a^2x^2-1)=2a^2+2a^2x^2,$$

$$a^2x^2 + 1 + a^4x^2 + a^2 - a(ax-1)^2 + \overline{a-1}\,.\sqrt{1+a^2}(a^2x^2-1) = 2a^2 + 2a^2x^2,$$

$$a^2x^2(a^2-1) - (a^2-1) - a(ax-1)^2 + \overline{a-1}\,.\sqrt{1+a^2}(a^2x^2-1) = 0,$$

$$(a^2-1)(a^2x^2-1) - a(ax-1)^2 + \overline{a-1}\,.\sqrt{1+a^2}\,.(a^2x^2-1) = 0;$$

$$\therefore\ ax - 1 = 0, \quad \text{or } x = \frac{1}{a}; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\text{I}).$$

Also $(a^2-1)(ax+1) - a(ax-1) + \overline{a-1}\,.\sqrt{1+a^2}(ax+1) = 0,$

or $\{a^2 - 1 + \overline{a-1}\,.\sqrt{1+a^2} - a\}ax = -\{a^2 - 1 + \overline{a-1}\,.\sqrt{1+a^2} + a\};$

$$\therefore\ -ax = \frac{a^2 + a - 1 + (a-1)\sqrt{1+a^2}}{a^2 - a - 1 + (a-1)\sqrt{1+a^2}},$$

$$= 1 + \frac{2a}{a^2 - a - 1 + (a-1)\sqrt{1+a^2}},$$

$$= 1 + \frac{2a(a^2 - a - 1 - \overline{a-1}\sqrt{1+a^2})}{(a^2-a-1)^2 - (a-1)^2(a^2+1)},$$

$$= 1 + \frac{2a(a^2 - a - 1 - \overline{a-1}\sqrt{1+a^2})}{a(4-3a)};$$

$$\therefore\ x = \frac{2}{a(4-3a)}\{\overline{a-1}\,.\sqrt{1+a^2} - (a^2-a-1)\} - \frac{1}{a}\ldots\ldots(\text{II}).$$

Ex. 23. $2x\sqrt{1-x^4} = a(1+x^4)$; find x.

Squaring, $4x^2 - 4x^6 = a^2 + 2a^2x^4 + a^2x^8.$

Dividing by a^2x^4, $\dfrac{4}{a^2x^2} - \dfrac{4x^2}{a^2} = \dfrac{1}{x^4} + 2 + x^4,$

$$\text{or}\quad -\frac{4}{a^2}\left(x^2 - \frac{1}{x^2}\right) = \left(x^2 + \frac{1}{x^2}\right)^2;$$

$$\therefore\ \left(x^2 + \frac{1}{x^2}\right)^2 + \frac{4}{a^2}\left(x^2 - \frac{1}{x^2}\right) = 0;$$

$$\left(x^2 - \frac{1}{x^2}\right)^2 + \frac{4}{a}\left(x^2 - \frac{1}{x^2}\right) = -4;$$

$$\left(x^2 - \frac{1}{x^2}\right)^2 + \frac{4}{a^2}\left(x^2 - \frac{1}{x^2}\right) + \left(\frac{2}{a^2}\right)^2 = \frac{4}{a^4} - 4;$$

$$x^2 - \frac{1}{x^2} + \frac{2}{a^2} = \pm\frac{2}{a^2}\sqrt{1-a^4};$$

$$\therefore\ x^2 - \frac{1}{x^2} = -\frac{2}{a^2}\{1 \mp \sqrt{1-a^4}\} = 2b^2, \text{ suppose; then}$$

$$x^4 - 2b^2x^2 = 1,$$

$$x^4 - 2b^2x^2 + b^4 = 1 + b^4,$$

$$x^2 = b^2 \pm \sqrt{1 + b^4};$$

$$\therefore\ x = \pm\sqrt{b^2 \pm \sqrt{1 + b^4}}.$$

But $b^2 = -\dfrac{1}{a^2}\{1 \mp \sqrt{1 - a^4}\}$,

$$1 + b^4 = 1 + \frac{1}{a^4}\{2 - a^4 \mp 2\sqrt{1 - a^4}\} = \frac{2}{a^4}\{1 \mp \sqrt{1 - a^4}\},$$

$$\therefore\ x = \pm\frac{1}{a}\left\{\sqrt{-1 \pm \sqrt{1 - a^4} \pm \sqrt{2\left(1 \mp \sqrt{1 - a^4}\right)}}\right\}.$$

Ex. 24. $\left(x - \dfrac{1}{3}\right)^2 - \dfrac{25}{9} = \dfrac{3x^2 + \dfrac{4}{9}}{2\left(x - \dfrac{1}{3}\right) + \sqrt{x\left(x - \dfrac{8}{3}\right)}}$; find x.

$$x^2 - \frac{2x}{3} - \frac{24}{9} = \frac{\left(3x^2 + \frac{4}{9}\right) \cdot \left\{2\left(x - \frac{1}{3}\right) - \sqrt{x\left(x - \frac{8}{3}\right)}\right\}}{4\left(x^2 - \frac{2x}{3} + \frac{1}{9}\right) - x^2 + \frac{8x}{3}} = \frac{\text{Num}^{\text{r}}.}{3x^2 + \frac{4}{9}},$$

$$= 2\left(x - \frac{1}{3}\right) - \sqrt{x\left(x - \frac{8}{3}\right)},$$

$$x^2 - \frac{8x}{3} + \sqrt{x^2 - \frac{8x}{3}} = 2,$$

$$x^2 - \frac{8x}{3} + \sqrt{x^2 - \frac{8x}{3}} + \frac{1}{4} = 2 + \frac{1}{4} = \frac{9}{4},$$

$$\sqrt{x^2 - \frac{8x}{3}} = \pm\frac{3}{2} - \frac{1}{2} = 1, \text{ or } -2.$$

$$x^2 - \frac{8x}{3} + \left(\frac{4}{3}\right)^2 = \frac{16}{9} + 1, \quad \text{or } x^2 - \frac{8x}{3} + \left(\frac{4}{3}\right)^2 = \frac{16}{9} - 2;$$

$$\therefore\ x - \frac{4}{3} = \pm\frac{5}{3}, \qquad = -\frac{2}{9}, \text{ which is impossible,}$$

and $x = \dfrac{4 \pm 5}{3} = 3$ or $-\dfrac{1}{3}$.

Ex. 25. $\dfrac{(n-1)(a^4 + a^2x^2 + x^4)}{(n+1)(a^4 - a^2x^2 + x^4)} = \left(2 - \dfrac{1}{n}\right) \cdot \left(\dfrac{ax}{a^2 - x^2}\right)^2$; find x.

$$\frac{a^4 + a^2x^2 + x^4}{a^4 - a^2x^2 + x^4} \cdot \left(\frac{a^2 - x^2}{ax}\right)^2 = \frac{2n-1}{n} \cdot \frac{n+1}{n-1},$$

$$\frac{\frac{a^2}{x^2}+1+\frac{x^2}{a^2}}{\frac{a^2}{x^2}-1+\frac{x^2}{a^2}}\cdot\left(\frac{a}{x}-\frac{x}{a}\right)^2=\frac{2n-1}{n}\cdot\frac{n+1}{n-1}=p \text{ suppose,}$$

$$\frac{\left(\frac{a}{x}-\frac{x}{a}\right)^2+3}{\left(\frac{a}{x}-\frac{x}{a}\right)^2+1}\cdot\left(\frac{a}{x}-\frac{x}{a}\right)^2=p. \quad \text{Let } \frac{a}{x}-\frac{x}{a}=y,$$

$$\text{then } \frac{y^2+3}{y^2+1}\cdot y^2=p,$$

$$y^4+3y^2=py^2+p,$$

$$y^4+(3-p)y^2=p; \text{ a quadratic equation for } y^2.$$

Let the value thus found for y be represented by c, then

$$\frac{a}{x}-\frac{x}{a}=c, \text{ or } a^2-x^2=cax, \text{ which gives } x.$$

Ex. 26. $\left.\begin{array}{r} x(bc-xy)=y(xy-ac)\ldots(1) \\ \text{and } xy(ay+bx-xy)=abc(x+y-c)\ldots(2)\end{array}\right\}$; find x and y.

$$\text{From (1)} \quad c(bx+ay)=xy(x+y),$$

$$c(bx+ay-xy)=xy(x+y-c)\ldots\ldots(3);$$

$$\therefore \frac{xy}{c}=\frac{abc}{xy}, \text{ dividing (2) by (3),}$$

$$\text{or } (xy)^2=abc^2,$$

$$\text{or } xy=c.\sqrt{ab};$$

$$\text{and from (1)} \quad \frac{x}{y}=\frac{xy-ac}{bc-xy}=\frac{c\sqrt{ab}-ac}{bc-c\sqrt{ab}},$$

$$=\frac{\sqrt{ab}-a}{b-\sqrt{ab}}=\frac{\sqrt{a}.(\sqrt{b}-\sqrt{a})}{\sqrt{b}.(\sqrt{b}-\sqrt{a})},$$

$$=\sqrt{\frac{a}{b}};$$

$$\therefore x^2=xy.\frac{x}{y}=c\sqrt{ab}.\sqrt{\frac{a}{b}}=ac,$$

$$\left.\begin{array}{r}\therefore x=\pm\sqrt{ac} \\ \text{and } y=\pm\sqrt{bc}\end{array}\right\}.$$

Ex. 27. $$\left.\begin{array}{r} x^3 + y(xy-1) = 0 \dots (1) \\ \text{and } y^3 - x(xy+1) = 0 \dots (2) \end{array}\right\}; \text{ find } x \text{ and } y.$$

From (1) $x^4 + x^2y^2 - xy = 0.$

From (2) $y^4 - x^2y^2 - xy = 0;$

$$\therefore\ x^4 - y^4 + 2x^2y^2 = 0,$$

$$x^4 + 2x^2y^2 + y^4 = 2y^4$$

$$x^2 + y^2 = \sqrt{2}\,.\,y^2$$

$$x^2 = (\sqrt{2} - 1)\,y^2$$

$$\frac{x^2}{y^2} = \sqrt{2} - 1, \quad (\alpha)$$

$$\therefore\ \frac{x}{y} = \sqrt{\sqrt{2} - 1}.$$

Again from (1) and (2) $\dfrac{x^3}{y^3} = \dfrac{y}{x}\cdot\dfrac{1 - xy}{1 + xy};$

$$\therefore\ \frac{1 - xy}{1 + xy} = \frac{x^4}{y^4} = 3 - 2\sqrt{2}, \text{ from } (\alpha);$$

$$\therefore \text{ by Art. 195, } xy = \frac{2\sqrt{2} - 2}{4 - 2\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\therefore\ xy \times \frac{x}{y} = x^2 = \frac{1}{\sqrt{2}}\sqrt{\sqrt{2} - 1};$$

$$\therefore\ x = \frac{\sqrt[4]{\sqrt{2} - 1}}{\sqrt[4]{2}} = \sqrt[4]{\frac{1}{2}(\sqrt{2} - 1)}.$$

$$\text{And } y = \frac{x}{\sqrt{\sqrt{2} - 1}} = \frac{1}{\sqrt[4]{2}\cdot\sqrt[4]{\sqrt{2} - 1}} = \frac{1}{\sqrt[4]{2(\sqrt{2} - 1)}}.$$

Ex. 28. $$\left.\begin{array}{l} xy = a(x + y) \dots (1) \\ xz = b\,(x + z) \dots (2) \\ yz = c(y + z) \dots (3) \end{array}\right\}; \text{ find } x, y, z.$$

From (1) $\dfrac{x + y}{xy} = \dfrac{1}{a},$

or $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{a}.$

From (2) $\frac{1}{x}+\frac{1}{z}=\frac{1}{b}$;

$$\therefore \frac{1}{y}-\frac{1}{z}=\frac{1}{a}-\frac{1}{b}.$$

From (3) $\frac{1}{y}+\frac{1}{z}=\frac{1}{c}$;

$$\therefore \frac{2}{y}=\frac{1}{a}+\frac{1}{c}-\frac{1}{b},$$

and $\frac{2}{z}=\frac{1}{b}+\frac{1}{c}-\frac{1}{a}$,

also $\frac{2}{x}=\frac{1}{a}+\frac{1}{b}-\frac{1}{c}$;

$$\therefore \left.\begin{aligned} x&=\frac{2abc}{ac+bc-ab} \\ y&=\frac{2abc}{ab+bc-ac} \\ z&=\frac{2abc}{ab+ac-bc} \end{aligned}\right\}.$$

Ex. 29. $x(x+y+z)=a^2$, $y(x+y+z)=b^2$, $z(x+y+z)=c^2$; find x, y, z.

Adding all together, $(x+y+z)(x+y+z)=a^2+b^2+c^2$;

$$\therefore x+y+z=\pm\sqrt{a^2+b^2+c^2};$$

$\therefore$ from (1) $x=\pm\frac{a^2}{\sqrt{a^2+b^2+c^2}}$,

...... (2) $y=\pm\frac{b^2}{\sqrt{a^2+b^2+c^2}}$,

...... (3) $z=\pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}$.

Ex. 30. $x(y+z)^2=1+a^3$; $x+y=\frac{3}{2}+z$; and $yz=\frac{3}{16}$; find x, y, z.

From (1) $(y+z)^2=\frac{1+a^3}{x}$;

$$\therefore (y-z)^2=\frac{1+a^3}{x}-4yz=\frac{1+a^3}{x}-\frac{3}{4},$$

But from (2) $y-z=\frac{3}{2}-x$,

$$\therefore \frac{1+a^3}{x} - \frac{3}{4} = \left(\frac{3}{2} - x\right)^2 = \frac{9}{4} - 3x + x^2;$$

$$\frac{1+a^3}{x} = x^2 - 3x + 3;$$

$$1 + a^3 = x^3 - 3x^2 + 3x,$$

$$a^3 = x^3 - 3x^2 + 3x - 1;$$

$$a = x - 1,$$

$$\therefore x = a + 1.$$

Hence, from (1) $(a+1)(y+z)^2 = 1 + a^3$,

$$\therefore (y+z)^2 = a^2 - a + 1;$$

$$\therefore y + z = \pm\sqrt{a^2 - a + 1} = \pm\sqrt{(a-1)^2 + a},$$

also $y - z = \frac{1}{2} - a$;

$$\therefore 2y = \frac{1}{2} - a \pm \sqrt{(a-1)^2 + a};$$

$$\therefore y = \frac{1}{4} - \frac{a}{2} \pm \frac{1}{2}\sqrt{(a-1)^2 + a};$$

and $z = \frac{a}{2} - \frac{1}{4} \pm \frac{1}{2}\sqrt{(a-1)^2 + a}$.

Ex. 31. $x(y+z) = a$, $y(x+z) = b$, $z(x+y) = c$; find x, y, z.

$$xy + xz = a,$$

$$xy + yz = b,$$

$$xz + yz = c;$$

$$\therefore 2xy + xz + yz = a + b;$$

$$2xy = a + b - c;$$

Similarly $2yz = b + c - a$,

and $2xz = a + c - b$;

$$\therefore \frac{2xy \cdot 2xz}{2yz} = 2x^2 = \frac{(a+b-c)\cdot(a+c-b)}{b+c-a}$$

$$\frac{2xy \cdot 2yz}{2xz} = 2y^2 = \frac{(a+b-c)\cdot(b+c-a)}{a+c-b}$$

$$\frac{2yz \cdot 2xz}{2xy} = 2z^2 = \frac{(b+c-a)\cdot(a+c-b)}{a+b-c};$$

$$\therefore\ x = \pm\sqrt{\frac{(a+b-c).(a+c-b)}{2(b+c-a)}},$$

$$y = \pm\sqrt{\frac{(a+b-c).(b+c-a)}{2(a+c-b)}},$$

$$z = \pm\sqrt{\frac{(b+c-a).(a+c-b)}{2(a+b-c)}}.$$

Ex. 32. $xy + n(x+y) = a$, $xz + n(x+z) = b$, $yz + n(y+z) = c$; find y, and z.

From (1), $(x+n)(y+n) = a + n^2$,

...... (2), $(x+n)(z+n) = b + n^2$,

...... (3), $(y+n)(z+n) = c + n^2$.

Multiply the 1st and 2nd together, and divide by the 3rd, of these uations, then

$$\frac{(x+n)^2(y+n)(z+n)}{(y+n)(z+n)} \text{ or } (x+n)^2 = \frac{(a+n^2)(b+n^2)}{(c+n^2)};$$

$$\therefore\ x + n = \pm\sqrt{\frac{(a+n^2)(b+n^2)}{c+n^2}},$$

$$\text{and}\quad x = -n \pm\sqrt{\frac{(a+n^2)(b+n^2)}{c+n^2}}.$$

Multiply the 1st and 3rd together, and divide by the 2nd; then we have

$$y + n = \pm\sqrt{\frac{(a+n^2)(c+n^2)}{b+n^2}},$$

$$\text{and}\quad y = -n \pm\sqrt{\frac{(a+n^2)(c+n^2)}{b+n^2}}.$$

Multiply the 2nd and 3rd together, and divide by the 1st; then

$$z + n = \pm\sqrt{\frac{(b+n^2)(c+n^2)}{a+n^2}},$$

$$\text{and}\quad z = -n \pm\sqrt{\frac{(b+n^2)(c+n^2)}{a+n^2}}.$$

Ex. 33. $x^a y^b = r$, and $x^c y^d = s$; find x and y.

Raising the first equation to to the dth power, and the second to the power,

$$x^m y^n = r^p,$$

and $$x^n y^n = s^q:$$

$$\therefore \frac{x^m}{x^n} = \frac{r^p}{s^q}, \text{ or } x^{m-n} = \frac{r^p}{s^q}:$$

$$\therefore x = \left(\frac{r^p}{s^q}\right)^{\frac{1}{m-n}}.$$

Similarly $$y = \left(\frac{s^q}{r^p}\right)^{\frac{1}{m-n}}.$$

Ex. 34. $x^{x+y} = y^{4a}$, and $y^{x+y} = x^a$; find x and y.

From first equation $x^{\frac{x+y}{4a}} = y$,

...... second ...... $x^{\frac{a}{x+y}} = y$;

$$\therefore x^{\frac{x+y}{4a}} = x^{\frac{a}{x+y}}, \text{ and } \frac{x+y}{4a} = \frac{a}{x+y};$$

$$(x+y)^2 = 4a^2, \text{ or } x+y = 2a;$$

$$\therefore x = y^2, \because x^{2a} = y^{4a};$$

$$\therefore y^2 + y = 2a.$$

From which it is found that $y = \frac{1}{2}\{-1 \pm \sqrt{8a+1}\}$; and then $x = \frac{1}{2}\{4a + 1 \mp \sqrt{8a+1}\}$.

Ex. 35. $$\left.\begin{aligned} &\frac{\sqrt{y^2+1}+1}{y} = \frac{\sqrt{x+9}+3}{\sqrt{x}} \text{......(1)} \\ \text{and } &x(y+1)^2 = 36\left(y^2 + \frac{16}{9}\right) \text{......(2)} \end{aligned}\right\}; \text{ find } x \text{ and } y.$$

From (1), multiplying the Numerator and Denominator of the first fraction by $\sqrt{y^2+1} - 1$, of the second by $\sqrt{x+9} - 3$, then multiplying the resulting equation by (1),

$$\frac{\sqrt{y^2+1}+1}{\sqrt{y^2+1}-1} = \frac{\sqrt{x+9}+3}{\sqrt{x+9}-3};$$

$$\therefore \sqrt{y^2+1} = \frac{\sqrt{x+9}}{3}, \text{ (Art. 195).}$$

$$y^2 + 1 = \frac{x+9}{9},$$

$$\therefore 9y^2 = x.$$

From (2), $9y^2(y^2+2y+1)=36y^3+64,$

$$9y^2(y^2-2y+1)=64,$$

$$3y(y-1)=\pm 8,$$

$$y^2-y=\pm\frac{8}{3},$$

$$y^2-y+\frac{1}{4}=\frac{1}{4}\pm\frac{8}{3}=\frac{35}{12},\ \text{or}\ -\frac{29}{12},$$

$$y-\frac{1}{2}=\pm\frac{1}{6}\sqrt{105},$$

$$\therefore\ y=\frac{1}{6}\{3\pm\sqrt{105}\}.$$

And $x=9y^2=\frac{1}{2}\{57\pm\sqrt{105}\}.$

Ex. 36.
$$\left.\begin{array}{l} 2x+\sqrt{x^2-y^2}=\dfrac{14}{y}\left\{\sqrt{\dfrac{x+y}{2}}+\sqrt{\dfrac{x-y}{2}}\right\} \\ \text{and}\ \left(\dfrac{x+y}{2}\right)^{\frac{3}{2}}+\left(\dfrac{x-y}{2}\right)^{\frac{3}{2}}=9 \end{array}\right\};\ \text{find } x \text{ and } y.$$

Assume $\frac{x+y}{2}=v$, and $\frac{x-y}{2}=w$,

then $2x=2(v+w)$,

also $x^2-y^2=4vw$, and $y=v-w$;

$\therefore$ by substitution in the first equation

$$2(v+w)+2\sqrt{vw}=\frac{14}{v-w}\cdot\{v^{\frac{1}{2}}+w^{\frac{1}{2}}\};$$

$$\text{or}\ v+w-\sqrt{vw}=\frac{7}{v^{\frac{1}{2}}-w^{\frac{1}{2}}};$$

$$\therefore\ v^{\frac{3}{2}}-w^{\frac{3}{2}}=7.$$

But $v^{\frac{3}{2}}+w^{\frac{3}{2}}=9$, from the second equation;

$$\therefore\ 2v^{\frac{3}{2}}=16,\quad \text{or}\ v=4,$$

$$\text{and}\ 2w^{\frac{3}{2}}=2,\quad \text{or}\ w=1;$$

$$\therefore\ x=5,\ \text{and}\ y=3.$$

By finding *all* the values of v and w which satisfy the equations $64=0$, $w^3-1=0$, by the method employed in Ex. 12, other values of nd y may be determined.

Ex. 37. $\left.\begin{aligned} \frac{y}{2x} + \frac{2}{3}\cdot\frac{y-\sqrt{x-1}}{y^2-2\sqrt{x^2-1}} &= \frac{\sqrt{x+1}}{x} \\ \text{and } \frac{1}{4}y^4 &= y^2x - 1 \end{aligned}\right\}$; find x and y.

From 2nd equation $y^4 - 4xy^2 = -4$;

$$y^4 - 4x\cdot y^2 + 4x^2 = 4x^2 - 4;$$

$$y^2 - 2x = \pm 2\sqrt{x^2-1};$$

$$\therefore y^2 = 2x \pm 2\sqrt{x^2-1};$$

$$\text{and } \therefore y = \pm(\sqrt{x+1} \pm \sqrt{x-1}) \ldots\ldots (1).$$

Substituting for y in the first equation, and taking the upper s in (1),

$$\frac{\sqrt{x+1}}{2x} + \frac{\sqrt{x-1}}{2x} + \frac{2}{3}\cdot\frac{\sqrt{x+1}}{2x} = \frac{\sqrt{x+1}}{x};$$

$$\therefore 3\sqrt{x-1} = \sqrt{x+1};$$

$$9x - 9 = x + 1,$$

$$8x = 10,$$

$$\therefore x = \frac{10}{8} = 1\tfrac{1}{4}.$$

$$\text{And } y = \sqrt{x+1} + \sqrt{x-1} = 2.$$

By taking the lower signs in (1) other values of x and y may obtained.

Ex. 38. $\left.\begin{aligned} x^2y - 4 &= 4yx^{\frac{1}{2}} - \frac{1}{4}y^3 \ldots(1) \\ \text{and } x^{\frac{3}{2}} - 3 &= x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}} - y^{\frac{1}{2}}) \ldots(2) \end{aligned}\right\}$; find x and y.

From (1), $y\left(x^2 + \frac{y^2}{4}\right) = 4yx^{\frac{1}{2}} + 4$;

$$\therefore y\left(x^2 + xy + \frac{y^2}{4}\right) = xy^2 + 4yx^{\frac{1}{2}} + 4;$$

$$\therefore y^{\frac{1}{2}}\left(x + \frac{y}{2}\right) = \pm(x^{\frac{1}{2}}y + 2).$$

Taking the + sign, $xy^{\frac{1}{2}} - x^{\frac{1}{2}}y = 2 - \frac{y^{\frac{3}{2}}}{2}$;

or $x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}}) = 2 - \frac{y^{\frac{3}{2}}}{2}$;

∴ by (2), $x^{\frac{3}{2}} - 3 = 2 - \frac{y^{\frac{3}{2}}}{2}$,

$2x^{\frac{3}{2}} + y^{\frac{3}{2}} = 10 \ldots\ldots\ldots(3).$

Also from (2), $3x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}}) = 3x^{\frac{3}{2}} - 9$

$= x^{\frac{3}{2}} - y^{\frac{3}{2}} + 1$, by (3);

∴ $x^{\frac{3}{2}} - y^{\frac{3}{2}} - 3x^{\frac{1}{2}}y^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}}) = -1$;

extract cube root, $x^{\frac{1}{2}} - y^{\frac{1}{2}} = -1$;

∴ $y^{\frac{1}{2}} = x^{\frac{1}{2}} + 1 \ldots\ldots\ldots(4).$

tuting in (3), $2x^{\frac{3}{2}} + x^{\frac{3}{2}} + 1 + 3x^{\frac{1}{2}}(x^{\frac{1}{2}}+1) = 10$,

$3x^{\frac{3}{2}} + 3x + 3x^{\frac{1}{2}} = 9$,

$x^{\frac{3}{2}} + x + x^{\frac{1}{2}} = 3$;

∴ $(x^{\frac{3}{2}}-1) + (x-1) + (x^{\frac{1}{2}}-1) = 0$,

or $(x^{\frac{1}{2}}-1)(x + 2x^{\frac{1}{2}} + 3) = 0$;

∴ $x^{\frac{1}{2}} - 1 = 0$, and $x = 1$

stituting in (4), $y^{\frac{1}{2}} = 1 + 1 = 2$; ∴ $y = 4$.

Also $x + 2x^{\frac{1}{2}} = -3$;

∴ $x + 2x^{\frac{1}{2}} + 1 = (x^{\frac{1}{2}}+1)^2 = -2$, which is impossible;

∴ $x = 1$, $y = 4$, are the required values.

x. 39. $$\left.\begin{aligned} a(1-xy) &= x\sqrt{1-y^2}\ldots(1) \\ \text{and } \sqrt{x}(1-xy) &= y - x\ldots\ldots\ldots(2) \end{aligned}\right\}; \text{ find } x \text{ and } y.$$

From (2), $x^{\frac{1}{2}} - x^{\frac{3}{2}}y = y - x$;

$$\therefore\ y = \frac{x^{\frac{1}{2}} + x}{1 + x^{\frac{3}{2}}};$$

$$1 - xy = 1 - \frac{x^{\frac{3}{2}} + x^2}{1 + x^{\frac{3}{2}}} = \frac{1 - x^2}{1 + x^{\frac{3}{2}}},$$

$$\text{and } 1 - y^2 = 1 - \frac{x + 2x^{\frac{3}{2}} + x^2}{1 + 2x^{\frac{3}{2}} + x^3}$$

$$= \frac{1 - x - x^2 + x^3}{(1 + x^{\frac{3}{2}})^2} = \frac{(1-x)^2(1+x)}{(1 + x^{\frac{3}{2}})^2}.$$

Substituting in (1), $a \cdot \frac{1-x^2}{1+x^{\frac{3}{2}}} = x \cdot \frac{1-x}{1+x^{\frac{3}{2}}} \sqrt{1+x}$;

$$\therefore\ a(1+x) = x\sqrt{1+x};\quad \text{also } 1-x=0, \text{ or } x=1 \ldots\ldots\ldots\ldots\text{(i)},$$

$$\therefore\ x = a\,.\sqrt{1+x};\quad \text{also } 1+x=0, \text{ or } x=-1 \ldots\ldots\ldots\ldots\text{(ii)},$$

$$x^2 = a^2 + a^2x,$$

$$x^2 - a^2x + \frac{a^4}{4} = \frac{a^4}{4} + a^2;$$

$$\therefore\ x = \frac{a^2 \pm a\sqrt{a^2+4}}{2} \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

Also $y = 1$, or -1, or $\pm\frac{a^2-1}{a^2+1}$, or $\frac{2a}{\sqrt{(a^2+1+a\sqrt{a^2+4})^2-1}-2a}$ ...(iv).

Ex. 40. $y^4 = x^2(ay-bx)$, and $x^2 = ax - by$; find x and y.

From first equation, $\frac{y^4}{x^2} = ay - bx$,

and by second equation, $x^2 = ax - by$;

$$\therefore\ \frac{y^4}{x^4} = \frac{ay-bx}{ax-by} = \frac{a\frac{y}{x}-b}{a-b\frac{y}{x}}.$$

Assume $\frac{y}{x} = t$, then $t^4 = \frac{at-b}{a-bt}$;

$$\therefore\ at^4 - bt^5 = at - b,$$

$$\text{or}\ \ at(t^3-1) - b(t^5-1) = 0;$$

$$\therefore\ at(t^2+t+1) - b(t^4+t^3+t^2+t+1) = 0;$$

$$\therefore\ t-1=0, \text{ or } t=1. \ldots\ldots\ldots \text{(1)}.$$

Also $bt^4 + (b-a)t^3 + (b-a)t^2 + (b-a)t + b = 0$;

$$\therefore\ b\left(t^2+\frac{1}{t^2}\right) - (a-b)\left(t+\frac{1}{t}\right) - (a-b) = 0,$$

$$\text{or}\ b\left(t+\frac{1}{t}\right)^2 - (a-b)\left(t+\frac{1}{t}\right) - (a+b) = 0;$$

$$\therefore \left(t+\frac{1}{t}\right)^2 - \frac{a-b}{b}\left(t+\frac{1}{t}\right) = \frac{a+b}{b};$$

$$\therefore\ t+\frac{1}{t} = \frac{a-b}{2b} \pm \sqrt{\frac{\overline{a-b}|^2}{4b^2} + \frac{a+b}{b}}$$

$$= \frac{a-b \pm \sqrt{a^2+2ab+5b^2}}{2b} = m, \text{ suppose};$$

Then $t^2 - mt = -1$,

$$\text{or } t = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} - 1} \ldots\ldots (2).$$

If from (1) $t = 1$, then $x = y = a - b$. .... (i).

If $t = \frac{m}{2} \pm \sqrt{\frac{m^2}{4} - 1}$, $x = a - bt = a - \frac{b}{2}(m \pm \sqrt{m^2 - 4})$, ... (ii).

and $y = tx = \frac{1}{2}\{a - \frac{b}{2}(m \pm \sqrt{m^2 - 4})\}(m \pm \sqrt{m^2 - 4})$ .... (iii).

$$\text{where } m = \frac{1}{2b}\{a - b \pm \sqrt{a^2 + 2ab + 5b^2}\}.$$

Ex. 41. $\left.\begin{array}{l} x^5 + y^5 = xy(x+y)^3 \ldots\ldots (1) \\ \text{and } y^2\sqrt{x} = (x+y)^{\frac{3}{2}} \ldots\ldots\ldots (2) \end{array}\right\}$; find x and y.

From (1), dividing by $x + y$,

$$x^4 - x^3y + x^2y^2 - xy^3 + y^4 = xy(x^2 + 2xy + y^2)$$
$$= x^3y + 2x^2y^2 + xy^3;$$
$$\therefore\ x^4 - 2x^3y - x^2y^2 - 2xy^3 + y^4 = 0;$$

dividing by x^2y^2, $\frac{x^2}{y^2} - 2 . \frac{x}{y} - 1 - 2 . \frac{y}{x} + \frac{y^2}{x^2} = 0$;

$$\frac{x^2}{y^2} + \frac{y^2}{x^2} - 2\left(\frac{x}{y} + \frac{y}{x}\right) = 1,$$

$$\left(\frac{x}{y} + \frac{y}{x}\right)^2 - 2\left(\frac{x}{y} + \frac{y}{x}\right) + 1 = 4,$$

$$\frac{x}{y} + \frac{y}{x} = 1 \pm 2 = 3, \text{ or } -1.$$

Taking the first value and multiplying by $\frac{x}{y}$,

$$\left(\frac{x}{y}\right)^2 + 1 = 3 . \frac{x}{y},$$

$$\left(\frac{x}{y}\right)^2 - 3 . \frac{x}{y} + \frac{9}{4} = \frac{5}{4}.$$

$$\therefore\ \frac{x}{y} = \frac{3 \pm \sqrt{5}}{2}.$$

But from (2), $xy^4 = (x + y)^3$,

$$\therefore\ \frac{3 \pm \sqrt{5}}{2} . y^5 = \left(\frac{x}{y} + 1\right)^3 . y^3 = \left(\frac{5 \pm \sqrt{5}}{2}\right)^3 . y^3,$$

$$\therefore\ y^2 = \frac{(5 \pm \sqrt{5})^3}{4(3 \pm \sqrt{5})} = \frac{50 \pm 20\sqrt{5}}{3 \pm \sqrt{5}} = \frac{50 \pm 10\sqrt{6}}{4};$$

$$\therefore\ y = \frac{5}{2}\sqrt{2 + \frac{2}{5}\sqrt{5}},$$

$$\text{and } x = \sqrt{\frac{50 \pm 20\sqrt{5}}{3 \mp \sqrt{5}}} = \frac{5}{2}\sqrt{10 \pm \frac{22}{5}\sqrt{5}}.$$

Ex. 42. $\left.\begin{aligned}(x^6 + 1)y &= (y^2 + 1)x^3 \ldots (1)\\ \text{and } (y^6 - 1)x &= 9(x^2 + 1)y^3 \ldots (2)\end{aligned}\right\}$; find x and y.

From (1), $$\left(x^3 + \frac{1}{x^3}\right)y = y^2 + 1;$$

$$\therefore\ x^3 + \frac{1}{x^3} = y + \frac{1}{y} \ldots\ldots\ldots (3).$$

From (2), $$y^3 + \frac{1}{y^3} = 9\left(x + \frac{1}{x}\right);$$

$$\text{or } \frac{1}{3}\left(y^3 + \frac{1}{y^3}\right) = 3\left(x + \frac{1}{x}\right);$$

$$\therefore\ \frac{1}{3}\left(y^3 + \frac{1}{y^3}\right) + y + \frac{1}{y} = x^3 + 3\left(x + \frac{1}{x}\right) + \frac{1}{x^3},$$

$$y^3 + \frac{1}{y^3} + 3\left(y + \frac{1}{y}\right) = 3\left(x + \frac{1}{x}\right)^3,$$

$$\therefore\ y + \frac{1}{y} = \left(x + \frac{1}{x}\right)\sqrt[3]{3}.$$

And $\therefore\ x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)\sqrt[3]{3}$, by (3);

dividing by $x + \frac{1}{x}$, $x + \frac{1}{x} = 0$, i. e. $x^2 = -1$, which is impossible.

Also $x^2 - 1 + \frac{1}{x^2} = \sqrt[3]{3}$;

$$x^2 + 2 + \frac{1}{x^2} = \sqrt[3]{3} + 3,$$

$$\therefore\ x + \frac{1}{x} = \sqrt{\sqrt[3]{3} + 3}.$$

Similarly $x - \frac{1}{x} = \sqrt{\sqrt[3]{3} - 1}$;

$$\therefore\ x=\frac{1}{2}\left\{\sqrt{\sqrt[3]{3}+3}+\sqrt{\sqrt[3]{3}-1}\right\},$$

$$\text{and } y+\frac{1}{y}=\left(x+\frac{1}{x}\right)\sqrt[3]{3}$$

$$=\sqrt[3]{3}\,.\,\sqrt{\sqrt[3]{3}+3}\,;$$

from which $y=\frac{1}{2}\left\{\sqrt[3]{3}\,.\,\sqrt{\sqrt[3]{3}+3}\pm\left(\sqrt{3\sqrt[3]{9}-1}\right)\right\}$.

Ex. 43. $$\left.\begin{array}{r} y+3\sqrt[3]{y}\left\{\sqrt[3]{a+bx}-\sqrt[3]{y}\right\}\sqrt[3]{a+bx}=2a\ldots(1)\\ \text{and }\ \dfrac{y-3\sqrt[3]{y}\,.\,\sqrt[3]{a^2-b^2x^2}}{\sqrt[3]{y}-\sqrt[3]{a+bx}}=2a\,.\,\sqrt[n]{a-bx}\ldots\ldots(2)\end{array}\right\}\text{; find } x \text{ and } y.$$

From (1), $y-3y^{\frac{2}{3}}(a+bx)^{\frac{1}{3}}+3y^{\frac{1}{3}}(a+bx)^{\frac{2}{3}}-(a+bx)=a-bx$;

$\therefore$ extracting the cube root,

$$\sqrt[3]{y}-\sqrt[3]{a+bx}=\sqrt[3]{a-bx}\ldots\ldots(3);$$

$$\text{or } \sqrt[3]{y}=\sqrt[3]{a+bx}+\sqrt[3]{a-bx}\,;$$

$$\therefore\ y=a+bx+a-bx+3\sqrt[3]{a^2-b^2x^2}\,.\,\{\sqrt[3]{a+bx}+\sqrt[3]{a-bx}\}$$

$$=2a+3\sqrt[3]{y}\,.\,\sqrt[3]{a^2-b^2x^2}\,;$$

$$\therefore\ y-3\sqrt[3]{y}\,.\,\sqrt[3]{a^2-b^2x^2}=2a\ldots\ldots(4).$$

Substituting (3) and (4) in (2),

$$\frac{2a}{\sqrt[3]{a-bx}}=2a\,.\,\sqrt[n]{a-bx},$$

$$(a-bx)^{\frac{1}{n}+\frac{1}{3}}=1,$$

$$a-bx=1;$$

$$\therefore\ x=\frac{a-1}{b},$$

$$\text{and } y=\{\sqrt[3]{2a-1}+1\}^3.$$

Ex. 44. $$\left.\begin{array}{r}(2+4xy-3x^2)^2=2-4x^2y^2+3x^4\ldots\ldots\ldots(1)\\ \text{and }\ 5y^2+\dfrac{27x^2}{32}=\dfrac{9xy}{2}+\dfrac{2xy+1}{x^2}\ldots\ldots\ldots\ldots(2)\end{array}\right\}\text{; find } x \text{ and } y.$$

From (1), $4+16xy+16x^2y^2-12x^2-24x^3y+9x^4=2-4x^2y^2+3x^4$

$$\text{or}\quad 2+16xy+20x^2y^2-12x^2-24x^3y+6x^4=0$$

$$1+8xy+10x^2y^2-6x^2-12x^3y+3x^4=0\,;$$

$$\therefore\ x^4-4x^3y+4x^2y^2=\frac{2x^2y^2}{3}+2x^2-2xy-\frac{2xy}{3}-\frac{1}{3}\,;$$

$$\therefore\ (x^2-2xy)^2-2(x^2-2xy)+1=\frac{2x^2y^2}{3}+\frac{4xy}{3}+\frac{2}{3}$$

$$=\frac{2}{3}(x^2y^2+2xy+1)\,;$$

$$\therefore\ x^2-2xy-1=\sqrt{\frac{2}{3}}\,.\,(xy+1)\,\ldots\ldots\ldots(3).$$

Again, from (2), $10y^2+\dfrac{27x^2}{16}=9xy+\dfrac{4xy+2}{x^2}\,;$

$$\therefore\ \frac{9x^4}{16}-3x^3y=-\frac{10x^2y^2}{3}+\frac{4xy}{3}+\frac{2}{3},$$

$$\text{or}\ \left(\frac{3x^2}{4}\right)^2-4xy\,.\,\frac{3x^2}{4}+4x^2y^2=\frac{2x^2y^2}{3}+\frac{4xy}{3}+\frac{2}{3}\,;$$

$$\therefore\ \frac{3x^2}{4}-2xy=\sqrt{\frac{2}{3}}(xy+1)\,\ldots\ldots\ldots(4)\,;$$

$$\frac{3x^2}{4}-2xy=x^2-2xy-1\,;$$

$$\therefore\ 3x^2=4x^2-4,$$

$$\therefore\ x=2.$$

Substituting in (4), $3-4y=(2y+1)\sqrt{\dfrac{2}{3}}\,;$

$$\therefore\ 2y=\frac{3-\sqrt{\frac{2}{3}}}{2+\sqrt{\frac{2}{3}}}=2-\frac{3}{2}\sqrt{\frac{2}{3}}\,;$$

$$\therefore\ y=1-\frac{1}{4}\sqrt{6}.$$

Ex. 45. $\left.\begin{array}{r}\sqrt[3]{\dfrac{27y^{\frac{3}{2}}-1}{x^3+3y^2-2xy^{\frac{3}{2}}}}=3\sqrt{\dfrac{x}{y}}\ \ldots(1)\\ \text{and}\ 3x^2+42xy+16y^2=4\sqrt{xy}\,(5x+11y)\ \ldots(2)\end{array}\right\}$; find x and y.

From (2), $3x^2 - 20x^{\frac{3}{2}}y^{\frac{1}{2}} + 42xy - 44x^{\frac{1}{2}}y^{\frac{3}{2}} + 16y^2 = 0$,

dividing by xy, $3\frac{x}{y} - 20\sqrt{\frac{x}{y}} + 42 - 44\sqrt{\frac{y}{x}} + 16\frac{y}{x} = 0.$

Let $\frac{x}{y} = t^2$, then $3t^2 - 20t + 42 - \frac{44}{t} + \frac{16}{t^2} = 0$;

$$\therefore\ 3t^4 - 20t^3 + 42t^2 - 44t + 16 = 0,$$

$$\therefore\ (3t-2)\,t^3 - (3t-2)\,6t^2 + (3t-2)\,10t - (3t-2)\,8 = 0,$$

$$\text{or } (3t-2)(t^3 - 6t^2 + 10t - 8) = 0;$$

$$\therefore\ 3t - 2 = 0, \text{ or } t^3 - 6t^2 + 10t - 8 = 0,$$

$$\text{i.e. } (t-4).(t^2 - 2t + 2) = 0,$$

$$\therefore\ t = \frac{2}{3};\ \text{ or } t - 4 = 0,\ \text{ i.e. } t = 4.\quad \begin{cases}\text{or } t^2 - 2t + 2 = 0, \\ \text{i.e. } (t-1)^2 \ = -1, \text{ which is impossible.}\end{cases}$$

Therefore, taking the first value of t, and substituting in (1)

$$\frac{\frac{729}{8}x^{\frac{3}{2}} - 1}{x^3 + \frac{243}{16}x^2 - \frac{27}{4}x^{\frac{5}{2}}} = 8;$$

$$\therefore\ 8x^3 - 54x^{\frac{5}{2}} + \frac{243}{2}x^2 - \frac{729}{8}x^{\frac{3}{2}} = -1.$$

Extracting the cube root, $2x - \frac{9}{2}x^{\frac{1}{2}} = -1$;

$$\therefore\ x - \frac{9}{4}x^{\frac{1}{2}} + \frac{81}{64} = \frac{49}{64},$$

$$x^{\frac{1}{2}} = \frac{9 \pm 7}{8} = 2, \text{ or } \frac{1}{4},$$

$$\left.\begin{aligned}\therefore\ x &= 4, \text{ or } \frac{1}{16}, \\ \text{and } y &= \frac{9}{4}x = 9, \text{ or } \frac{9}{64}.\end{aligned}\right\}$$

Taking the second value of t, other values of x and y remain to be termined.

Ex. 46. $\left\{\frac{ab}{2}(b-c)^2 + (b^2+c^2)cx\right\} \times \sqrt{\frac{a^2}{4}(b-c)^4 + (b^2+c^2)^2x^2}$

$= \left\{\frac{ab}{2}(b-c)^2 - (b^2+c^2)cx\right\} \times \sqrt{\frac{a^2}{4}(b+c)^4 + (b^2+c^2)^2x^2}$; find x.

Let $b+c=m$, and $b-c=n$, so that $b^2+c^2=\frac{1}{2}(m^2+n^2)$,

$$\text{then}\quad \frac{m^2ab+(m^2+n^2)cx}{n^2ab-(m^2+n^2)cx}=\sqrt{\frac{m^4a^2+(m^2+n^2)^2x^2}{n^4a^2+(m^2+n^2)^2x^2}},$$

$$\frac{m^4a^2b^2+2m^2(m^2+n^2)abcx+(m^2+n^2)^2c^2x^2}{n^4a^2b^2-2n^2(m^2+n^2)abcx+(m^2+n^2)^2c^2x^2}=\frac{m^4a^2+(m^2+n^2)^2x^2}{n^4a^2+(m^2+n^2)^2x^2},$$

$$\frac{(m^4-n^4)a^2b^2+2(m^2+n^2)^2abcx}{n^4a^2b^2-2n^2(m^2+n^2)abcx+(m^2+n^2)^2c^2x^2}=\frac{(m^4-n^4)a^2}{n^4a^2+(m^2+n^2)^2x^2},$$

$$\frac{(m^2-n^2)ab^2+2(m^2+n^2)bcx}{n^4a^2b^2-2n^2(m^2+n^2)abcx+(m^2+n^2)^2c^2x^2}=\frac{(m^2-n^2)a}{n^4a^2+(m^2+n^2)^2x^2},$$

$$(m^2-n^2)n^4a^3b^2+(m^4-n^4)(m^2+n^2)ab^2x^2+2(m^2+n^2)n^4a^2bcx+2(m^2+n^2)^3bcx^3$$
$$=(m^2-n^2)n^4a^3b^2-2(m^4-n^4)n^2a^2bcx+(m^4-n^4)(m^2+n^2)ac^2x^2,$$

$$(m^4-n^4)ab^2x+2n^4a^2bc+2(m^2+n^2)^2bcx^2=(m^4-n^4)ac^2x-2(m^2-n^2)n^2a^2bc,$$

$$2(m^2+n^2)^2bcx^2+(m^4-n^4)a(b^2-c^2)x+2m^2n^2a^2bc=0,$$

$$\left.\begin{array}{c}(m^2+n^2)^2x^2+2(m^2+n^2)mnax+m^2n^2a^2=0,\\ (m^2+n^2)x+mna=0,\end{array}\right\}\begin{array}{l}\because\ mn=b^2-c^2\\ \text{and } bc=\frac{1}{4}(m^2-n^2)\end{array}$$

$$\therefore\ x=-\frac{mna}{m^2+n^2}=\frac{a}{2}\cdot\frac{c^2-b^2}{c^2+b^2}.$$

$$\text{Ex. 47.}\quad \left.\begin{array}{r}\dfrac{3+2x^2-4x^4}{x^2-1}=y^2(1-2y^2)\\ \text{and } (2x^2-1)(2y^2-1)=3\end{array}\right\};\ \text{find } x \text{ and } y.$$

$$\left.\begin{array}{l}\text{Write } u \text{ for } x^2\\ \text{and } v \text{ for } y^2\end{array}\right\},\quad \begin{array}{r}\text{then from (2), } 4uv-2(u+v)=2,\\ \text{or } 2uv-(u+v)=1,\ \ldots\ldots\ (a).\end{array}$$

$$\begin{aligned}\text{From (1),}\quad 4u^2-2u-3&=v(2v-1)(u-1)\\ &=v(2uv-u-2v+1)\\ &=v(2-v),\ \text{by } (a)\\ &=2v-v^2;\\ \therefore\ 4u^2-3&=2(u+v)-v^2\\ &=4uv-2-v^2,\ \text{by } (a),\end{aligned}$$

$$4u^2 - 4uv + v^2 = 1,$$
$$2u - v = \pm 1,$$
$$\therefore\ v = 2u \mp 1 \ \ldots\ldots\ (\beta).$$

Taking the upper sign, and substituting in (α),

$$4u^2 - 2u - u - 2u + 1 = 1,$$
$$\text{or } 4u^2 = 5u,\ \therefore\ u = \frac{5}{4},\ \text{or } 0 \ \ldots\ldots\ \text{(i)}.$$

Taking the lower sign, and substituting in (α),

$$4u^2 + 2u - u - 2u - 1 = 1,$$
$$4u^2 - u = 2,$$
$$4u^2 - u + \left(\frac{1}{4}\right)^2 = \frac{1}{16} + 2 = \frac{33}{16}$$
$$2u - \frac{1}{4} = \pm\frac{\sqrt{33}}{4},\ \therefore\ u = \frac{1 \pm \sqrt{33}}{8} \ \ldots\ldots\ldots\ \text{(ii)}.$$

Hence from (i), $x = \pm\frac{1}{2}\sqrt{5}$, or 0; and $y = \pm\frac{1}{2}\sqrt{6}$; or $\pm\sqrt{-1}$.

from (ii), $x = \pm\frac{1}{2}\sqrt{\frac{1 \pm \sqrt{33}}{2}}$; and $y = \pm\frac{1}{2}\sqrt{5 \pm \sqrt{33}}$.

PROBLEMS.

In reducing a Problem to an Equation, the course to be pursued
stated in Art. 199; but much depends here, as in the solution of equa-
ns, upon a practical acquaintance with particular artifices, by which the
st *convenient* unknown quantities are assumed, and the problem most
ily translated into algebraical language.

The general question always is, having certain known quantities,
resented by given symbols, and one or more other unknown quantities,
resented by one or more of the letters x, y, z, &c., to connect the known
d unknown symbols together by the conditions of the problem, so as to
duce as many *independent* equations as there are unknown quantities.

There is also one general property of a large class of such problems,.
z., that the increase or decrease, the selling or buying, &c., is after a
iform rate. Thus, if A is said to perform a piece of work in a days,
is supposed to work equally every day. If A is said to travel p miles
q days, he is supposed to travel one uniform distance each day. And

so on, unless the contrary be expressed. So that the following Rule is of constant application, seeing that uniform increase or decrease of every sort may be represented by uniform motion:—

Rule. If v represent the space described by a body moving *uniformly* in 1 unit of time, (whether it be 1 second, 1 hour, or any other known unit), and s the space described by the same body in t such units of time, then $s = tv$.

Also $v = \frac{s}{t}$, and $t = \frac{s}{v}$; both of which forms are frequently required.

Thus, if A travels p miles in q days, then the distance (v) travelled in 1 day $= \frac{\text{whole distance } (s)}{\text{number of days } (t)} = \frac{p}{q}$; and the number of days $= \frac{\text{whole distance}}{\text{distance per day}}$.

The following Problems are added here as differing in some material respect from those in the text:—

Prob. 1. In the year 1830 A's age was 50 and B's 35. Required the year in which A is twice as old as B.

Let $1830 \pm x$ be the year required.

$$\text{Then} \quad 50 \pm x = 2(35 \pm x),$$

$$\text{or} \quad 50 \pm x = 70 \pm 2x;$$

$$\therefore \pm x = -20;$$

$\therefore$ the year required is 1810.

Prob. 2. In what proportions must substances of "specific gravities" a and b be mixed, so that the "specific gravity" of the mixture may be c?

Def. By the "specific gravity" of a body is meant the number of times which its weight is of the weight of an equal bulk of water.

To 1 cubic foot of the first substance let x cubic feet of the second be added.

Then, since 1 cubic foot of the first weighs a cubic feet of water,

and x .... feet ...... second ...... bx ....

$\therefore$ the whole $1 + x$ cubic feet of mixture weighs $a + bx$ cubic feet of water.

But since c is the specific gravity of the mixture, the weight of $1 + x$ cubic feet is $c(1 + x)$ cubic feet of water,

$$\therefore a + bx = c(1 + x)$$

$$= c + cx;$$

$$\therefore x = \frac{a - c}{c - b};$$

that is, for every cubic foot of the substance whose specific gravity is a there must be $\frac{a - c}{c - b}$ cubic feet of the substance whose specific gravity is b.

Prob. 3. From a vessel of wine containing a gallons b gallons are drawn off, and the vessel is filled up with water. Find the quantity of wine remaining in the vessel, when this has been repeated n times.

Let $x_1, x_2, x_3, \ldots x_n$ be the number of gallons of wine remaining in the vessel after $1, 2, 3, \ldots n$ drawings off respectively.

(1) Then $x_1 = a - b$.

(2) $x_2 = a - b -$ quantity of wine in b gallons of first mixture.

Now, a gallons contain $a - b$ of wine;

$\therefore$ 1 gallon contains $\dfrac{a-b}{a}$ ....

and b gallons contain $b \cdot \dfrac{a-b}{a}$ ....

$$\therefore\ x_2 = a - b - b \cdot \frac{a-b}{a} = \frac{(a-b)^2}{a}.$$

(3) $x_3 = \dfrac{(a-b)^2}{a} -$ quantity of wine in b gallons of second mixture.

But a gallons contain $\dfrac{(a-b)^2}{a}$ of wine;

$\therefore$ 1 gallon contains $\dfrac{(a-b)^2}{a^2}$ ....

and b gallons contain $b \cdot \dfrac{(a-b)^2}{a^2}$ ....

$$\therefore\ x_3 = \frac{(a-b)^2}{a} - \frac{b \cdot (a-b)^2}{a^2} = \frac{(a-b)^3}{a^2}.$$

And so on, for each succeeding mixture; so that, generally,

$$x_n = \frac{(a-b)^n}{a^{n-1}}.$$

Prob. 4. The advance of the hour-hand of a watch before the minute-hand is measured by $15\frac{2}{3}$ of the minute divisions; and it is between 9 and 10 o'clock. Find the exact time indicated by the watch.

Let x = number of minutes past 9 o'clock; then since the minute-hand goes 12 times as fast as the hour-hand,

$45 + \dfrac{x}{12}$ = number of minute divisions the hour-hand is past 9,

and $45 + \dfrac{x}{12} - x$ = distance in minutes between hour-hand and minute-hand.

$= 15\frac{2}{3}$, by the question,

$$\therefore\ \frac{11x}{12} = 29\tfrac{1}{3} = \frac{88}{3},$$

$$\therefore\ x = 32.$$

Hence the time required is 28 minutes before 10 o'clock.

Prob. 5. In comparing the rates of a watch and a clock, it was observed on one morning, when it was 12^h. by the clock, that the watch was at $11^h.59^m.49^s$; and two mornings after, when it was 9^h. by the clock, the watch was at $8^h.59^m.58^s$. The clock is known to gain $0{\cdot}1^s$ in 24 hours, find the gaining rate of the watch.

On the 1st morning the watch is behind the clock 11^s; and after 45 hours it is only 2^s behind; ∴ the watch gains *upon the clock* to the amount of 9^s in 45 hours, or $\frac{1}{5}$ of a second per hour.

Let x be the gaining rate of the watch per hour; then since $\frac{0{\cdot}1}{24}$ is the gaining rate of the clock, $x - \frac{0{\cdot}1}{24}$ is the gain of the watch upon the clock *per hour*.

$$\therefore\ x - \frac{0{\cdot}1}{24} = \frac{1}{5},$$

$$\text{or } x = \frac{1}{240} + \frac{1}{5} = \frac{49}{240}.$$

Hence the gaining rate of the watch is $4{\cdot}9^s$ in 24 hours.

Prob. 6. If A and B together can perform a piece of work in a days, A and C together the same in b days, and B and C together in c days; find the time in which each can perform the work separately.

Let w represent the work, and x, y, z, the times in which A, B, C, can separately do it.

$$\text{Then } \frac{w}{a} = A\text{'s daily work} + B\text{'s daily work} \ldots\ldots\ldots (1),$$

$$\frac{w}{b} = A\text{'s} \ldots\ldots\ldots\ldots\ldots + C\text{'s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$\frac{w}{c} = B\text{'s} \ldots\ldots\ldots\ldots\ldots + C\text{'s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3);$$

$$\therefore\ \frac{w}{a} - \frac{w}{b} = B\text{'s} \ldots\ldots\ldots\ldots\ldots - C\text{'s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4);$$

and adding (3) and (4) $\frac{w}{a} + \frac{w}{c} - \frac{w}{b} = 2\,.\,B\text{'s daily work}, = 2\,.\,\frac{w}{y}$;

$$\therefore\ y = \frac{2abc}{ab + bc - ac}.$$

$$\text{Similarly, } x = \frac{2abc}{ac + bc - ab}; \quad \text{and } z = \frac{2abc}{ab + ac - bc}.$$

To shew that the denominators of these fractions are *necessarily* ositive:

By the Prob. B alone could not perform w in a days,

$\therefore$ B alone .... $\frac{w}{a}$ in 1 day.

C alone .... w in b days,

$\therefore$ C alone .... $\frac{w}{b}$ in 1 day;

$\therefore$ B and C together .... $\frac{w}{a}+\frac{w}{b}$ in 1 day,

d $\therefore$ B and C together .... $c\,.\left(\frac{w}{a}+\frac{w}{b}\right)$ in c days.

But B and C together can perform w in c days;

$$\therefore\ c\left(\frac{w}{a}+\frac{w}{b}\right)>w,\ \text{ or }\ \frac{ac+bc}{ab}>1,$$

$$\text{and } \therefore\ ac+bc>ab.$$

Similarly it may be shewn that $ab+bc>ac$, and $ab+ac>bc$.

PROB. 7. The fore-wheel of a coach makes 6 revolutions more than e hind-wheel in going 120 yards; but if the circumference of each heel be increased 1 yard, the fore-wheel will make only 4 revolutions ore than the hind-wheel in the same distance. Required the circumrence of each wheel.

Let x = circumf. of the hind-wheel, in yards;

y = ... fore-wheel ....

then $\frac{120}{x}$ = number of revolutions by former in 120 yards;

$\frac{120}{y}$ = ... latter ....

$$\therefore\ \frac{120}{x}+6=\frac{120}{y},$$

$$\text{or } 20x-20y=xy\ldots\ldots\ldots\ldots(1).$$

Again, $\frac{120}{x+1}+4=\frac{120}{y+1}$, on the 2nd supposition,

$$\text{or }\ 30(y+1)+(x+1)(y+1)=30(x+1);$$

$$\therefore\ 29x-31y=xy+1\ldots\ldots\ldots(2).$$

Subtr. (1) from (2), $9x - 11y = 1$,

$$\therefore\ 9x = 11y + 1 \ldots\ldots\ldots\ldots (3).$$

But from (1), $20 \times 9x - 20 \times 9y = 9xy$,

$$\therefore\ 20(11y+1) - 180y = 11y^2 + y,$$

$$\text{or}\quad 11y^2 - 39y = 20,$$

$$y^2 - \frac{39}{11}y + \left.\overline{\frac{39}{22}}\right|^2 = \frac{1521}{(22)^2} + \frac{20}{11} = \frac{2401}{(22)^2},$$

$$\therefore\ y = \frac{39 \pm 49}{22} = 4, \quad \text{or} -\frac{5}{11},$$

$$\text{and } x = \frac{11y+1}{9} = 5, \quad \text{or} -\frac{5}{9}:$$

∴ the circumferences of the wheels are 4 and 5 yards respectively.

Prob. 8. Find two numbers whose product is equal to the difference of their squares, and the sum of their squares equal to the difference of their cubes.

Let x and xy be the two numbers,

$$\left.\begin{aligned} \text{then } x^2y &= x^2y^2 - x^2 \ldots\ldots (1) \\ \text{and } x^2y^2 + x^2 &= x^3y^3 - x^3 \ldots\ldots (2) \end{aligned}\right\},$$

$$\text{From (1),}\quad y = y^2 - 1,$$

$$\therefore\ y^2 - y + \frac{1}{4} = \frac{5}{4},$$

$$\text{and } y = \frac{1}{2}(\sqrt{5} + 1).$$

$$\text{From (2),}\quad y^2 + 1 = xy^3 - x,$$

$$\therefore\ x = \frac{y^2+1}{y^3-1} = \frac{y+2}{2y} = \frac{1}{2} + \frac{1}{y},$$

$$= \frac{1}{2} + \frac{2}{\sqrt{5}+1} = \frac{1}{2} + \frac{\sqrt{5}-1}{2} = \frac{1}{2}\sqrt{5}.$$

$$\therefore \text{ the required numbers are } \frac{1}{2}\sqrt{5}, \text{ and } \frac{1}{4}(5 + \sqrt{5}).$$

Prob. 9. There are four numbers in Arithmetical Progression. The sum of the two extremes is 8, and the product of the means is 15. What are the numbers?

Let $x - 3y$, $x - y$, $x + y$, $x + 3y$, be the numbers;

then by the question, $x - 3y + x + 3y = 8$,

$$\therefore\ x = 4.$$

$$\text{Also } (x-y(x+y)=15,$$

$$\text{or . } x^2-y^2=15,$$

$$\therefore\ 16-y^2=15, \text{ and } y=1,$$

∴ the numbers are 1, 3, 5, 7.

Prob. 10. There are three numbers in Geometrical Progression, hose product is 64, and sum 14. What are the numbers?

Let $\frac{x}{y}$, x, xy be the numbers; then by the question,

$$\frac{x}{y}.x.xy \text{ or } x^3=64, \ \therefore\ x=4.$$

$$\text{Also } \frac{x}{y}+x+xy=14, \text{ or } \frac{1}{y}+1+y=\frac{14}{x}=\frac{7}{2},$$

$$\therefore\ y^2-\frac{5}{2}y+\left(\frac{5}{4}\right)^2=\frac{25}{16}-1=\frac{9}{16},$$

$$y=\frac{5\pm3}{4}=2, \text{ or } \frac{1}{2}.$$

∴ the numbers are 2, 4, 8, or 8, 4, 2.

Prob. 11. Two labourers A and B, whose rates of working are as to 5, were employed to dig a ditch; A worked 12 hours and B 10 hours day: B being called away, A worked one day alone in order to comete the work: when they were paid, B received as many pence more an A as the number of days they worked together. Now, had B been lled away a day sooner, A would have received 3*s.* 11*d.* more than B the conclusion of the work. Required their respective daily wages, on pposition that the payment to each was in proportion to the work permed.

Let x be the number of days they worked together;

$3w$, and $5w$ the work per hour of A and B respectively;

$$\therefore\ 36w = A\text{'s daily work},$$

$$50w = B\text{'s} \ldots\ldots\ldots\ldots\ldots$$

$$\text{let then } 36y = A\text{'s daily wages, in pence},$$

$$50y = B\text{'s} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\text{and we have } 50y\times x-36y(x+1)=x,$$

$$\text{or } 14xy-36y=x \ldots\ldots\ldots\ldots(1).$$

Again, on the second supposition, A and B work $x-1$ days together,

$$\therefore \text{ the work done in that time} = 86w(x-1),$$

$$\text{but the whole work} = 86wx + 36w;$$

$$\therefore \text{ work left to be done by } A = 122w;$$

$$\therefore A \text{ works } x - 1 + \frac{122w}{36w} \text{ days,}$$

$$\text{or } x + 2 + \frac{7}{18} \text{ days:}$$

$$\text{Hence } \left(x + 2 + \frac{7}{18}\right) 36y - (x-1)\,50y = 47,$$

$$\text{or } 136y - 14xy = 47 \text{ ....(2).}$$

$$\text{From (1), } x = \frac{36y}{14y-1};$$

$$\therefore 136y - \frac{14 \times 36y^2}{14y-1} = 47,$$

$$14 \times 136y^2 - 136y - 14 \times 36y^2 = 47 \times 14y - 47,$$

$$1400y^2 - 794y = -47,$$

$$y^2 - \frac{794}{1400}y + \overline{\frac{397}{1400}}\Big|^2 = \frac{91809}{(1400)^2};$$

$$\therefore y = \frac{397 \pm 303}{1400} = \frac{1}{2}, \text{ or } \frac{47}{700};$$

$\therefore$ A's daily wages $= 36 \times \frac{1}{2} = 18d. = 1s.\ 6d.$

and B's .... $= 50 \times \frac{1}{2} = 25d. = 2s.\ 1d.$

Prob. 12. It was calculated that, if the gross revenue of a state were increased in the proportion of $2\frac{1}{4} : 1$, after deducting the interest of the national debt and the cost of collection, (the latter of which varies as the square root of the sum collected), the available income would be increased in the proportion of $3\frac{16}{81} : 1$. If, on the other hand, the gross revenue were diminished in the proportion of $1\frac{7}{9} : 1$, the available income would be reduced in the proportion of $7\frac{3}{4} : 1$, and would in fact amount to only 4 millions. Find the amount of the revenue, and the interest of the debt.

$$\left.\begin{aligned} \text{Let } x &= \text{the gross revenue,} \\ y &= \text{the interest on the debt,} \\ z &= \text{the expence of collection,} \end{aligned}\right\} \textit{in millions of pounds sterling.}$$

hen $\frac{9x}{4}$ = increased revenue, and expence of collecting : $z :: \sqrt{x} : \sqrt{\frac{9x}{4}}$;

$$\therefore \text{ expence of collecting } = \frac{3z}{2};$$

$\frac{9x}{16}$ = diminished revenue, and expence of collecting $= \frac{3z}{4}$;

$$\therefore \frac{9x}{16} - y - \frac{3z}{4} = 4 \ldots\ldots\ldots\ldots (1);$$

$$\text{Also} \quad x \quad y - z : \frac{9x}{16} - y - \frac{3z}{4} :: 7\tfrac{3}{4} : 1 :: 31 : 4,$$

$$\therefore \text{ by } (1), \quad x - y - z = 31 \ldots\ldots\ldots\ldots\ldots (2).$$

$$\text{Again,} \quad x - y - z : \frac{9x}{4} - y - \frac{3z}{2} :: 1 : 3\tfrac{16}{31} :: 31 : 109;$$

$$\therefore \text{ by } (2), \quad \frac{9x}{4} - y - \frac{3z}{2} = 109 \ldots. (3).$$

$$\left.\begin{array}{l} \text{Subtracting (1) from (2)} \quad \frac{7x}{16} - \frac{z}{4} = 27, \\ \ldots\ldots\ldots\ldots\ldots \text{ (2) from (3)} \quad \frac{5x}{4} - \frac{z}{2} = 78, \end{array}\right\};$$

$$\therefore \frac{5x}{4} - \frac{7x}{8} = 78 - 54 = 24,$$

$$\frac{3x}{8} = 24, \text{ and } x = 64.$$

$$\text{Hence } \frac{z}{4} = \frac{7x}{16} - 27 = 28 - 27 = 1, \text{ and } \therefore z = 4.$$

$$\text{Also } y = x - z - 31 = 60 - 31 = 29.$$

Prob. 13. A steam-boat sets out from London 3 miles behind a ıerry, and having got to the same distance a-head it overtakes a barge *ating* down the stream, and reaches Gravesend $1\frac{1}{2}$ hours afterwards. aving waited to land the passengers $\frac{1}{5}$th of the time of coming down, starts to return, and meets the wherry in $\frac{3}{4}$ of an hour, the barge ing then $5\frac{1}{4}$ miles a-head of the steam-boat, and arrives at London in e same time that the wherry was in coming down. Find the distance tween London and Gravesend, and the rate of each vessel.

Let x = rate of the boat, y = rate of the wherry, t = rate of the tide, ıt is, of the barge; then $\because$ boat's speed against the tide = wherry's eed with it, $x - t = y + t$, $\therefore t = \frac{x - y}{2}$.

Hence $x+\frac{x-y}{2}$ or $\frac{3x-y}{2}=$ boat's speed down,

$x-\frac{x-y}{2}$ or $\frac{x+y}{2}=$ .... up,

$\frac{6}{x-y}=$ time before the boat overtakes the barge;

$\therefore\ \frac{6}{x-y}+\frac{3}{2}=$ whole time of boat down,

$\frac{6}{5(x-y)}+\frac{3}{10}=$ time for landing the passengers;

$\therefore\ \frac{6}{5(x-y)}+\frac{3}{10}+\frac{3}{4}+\frac{3}{2}$ or $\frac{6}{5(x-y)}+\frac{51}{20}=$ interval of time between the boat passing the barge, and meeting the wherry in returning, in which time the barge moves over $\frac{x-y}{2}\left\{\frac{6}{5(x-y)}+\frac{51}{20}\right\}$, or $\frac{3}{5}+\frac{51}{40}(x-y)$ miles, and the boat has come up $\frac{3}{4}\cdot\frac{x+y}{2}$ miles,

$\therefore\ \frac{3}{4}\cdot\frac{x+y}{2}+\frac{3}{5}+\frac{51}{40}(x-y)+\frac{21}{4}$ ($=$ distance from Gravesend when boat passed the barge down) $=\frac{3}{2}\cdot\frac{3x-y}{2}$, from which equation we get,

$$4x+y=39 \text{ .... } (1).$$

Again $\frac{8\frac{1}{4}}{y}=$ time wherry takes to get from 3 miles behind to $5\frac{1}{4}$ miles a-head of the barge,

$$\therefore\ \frac{33}{4y}=\frac{6}{5(x-y)}+\frac{51}{20} \text{ .... } (2),$$

from which two equations (1) and (2), $x=9$, $y=3$.

Hence also distance from London to Gravesend = time down × speed,

$$=\frac{5}{2}\times 12=30 \text{ miles.}$$

PROB. 14. *A* and *B* travelled on the same road and at the same rate to London. At the 50th mile-stone from London *A* overtook a flock of geese, which travelled at the rate of 3 miles in 2 hours; and 2 hours afterwards he met a stage-waggon which travelled at the rate of 9 miles in 4 hours. *B* overtook the flock of geese at the 45th mile-stone from London, and met the stage-waggon 40 minutes before he came to the 31st mile-stone. Where was *B* when *A* reached London?

Since A and B travel in the same direction on the same road and at the same rate, the distance between them is always the same.

Let x = the number of miles per hour of A's and B's travelling.

Then, since the places at which A and B overtake the geese are 5 miles apart, which the geese travel over in $5 \div \frac{3}{2}$ or $\frac{10}{3}$ hours; therefore in that time A has moved forward $\frac{10x}{3}$ miles;

$$\therefore\ \frac{10x}{3} - 5 = \text{distance in miles between } A \text{ and } B \ \ldots\ldots\ldots\ldots\ (1).$$

Again, A met the waggon $50 - 2x$ miles from London,

$$B \ldots\ldots\ldots\ldots\ldots\ldots\ 31 + \frac{2x}{3}\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$\therefore$ distance the waggon travelled between the meetings is $\frac{8x}{3} - 19$ miles; and the time elapsed between A and B meeting the waggon

$$= \left(\frac{8x}{3} - 19\right) \div \frac{9}{4} \text{ or } \frac{4}{9}\left(\frac{8x}{3} - 19\right) \text{ hours.}$$

During this time A has moved forward $\frac{4}{9}\left(\frac{8x}{3} - 19\right) x$ miles;

$$\therefore\ \text{distance between } A \text{ and } B = \frac{8x}{3} - 19 + \frac{4x}{9}\left(\frac{8x}{3} - 19\right) \text{ miles} \ldots\ldots (2);$$

$$\text{equating (1) and (2), } \frac{8x}{3} - 19 + \frac{4x}{9}\left(\frac{8x}{3} - 19\right) = \frac{10x}{3} - 5,$$

$$\frac{4x}{9}\left(\frac{8x}{3} - 19\right) = \frac{2x}{3} + 14,$$

$$16x^2 - 1026 = 9x + 189;$$

$$\therefore\ 16x^2 - 9x = 1215.$$

$$\therefore\ 16x^2 - 9x + \left(\frac{9}{8}\right)^2 = \frac{81}{64} + 1215 = \frac{77841}{64};$$

$$\therefore\ 4x - \frac{9}{8} = \frac{279}{8};$$

$$4x = \frac{288}{8} = 36;$$

$$\therefore\ x = 9,$$

and $\frac{10x}{3} - 5 = 25$, the required distance in miles of B from London.

Prob. 15. Fine gold chains are manufactured at Venice, and are sold at so much per braccio, a braccio being a measure containing about two feet English. When there are 90 links in an inch, the value of the workmanship of a braccio is equal to the whole value of a braccio when there are but 30 links in an inch; and the whole value of the braccio in the former case is equal to three-times the difference between the cost of the material and workmanship of a braccio in the latter, together with $4\frac{4}{9}$ francs. Supposing that the workmanship in each braccio varies as the number of links in an inch, and the weight of metal varies inversely as the square of that number, find the values of the material and workmanship in a braccio of each of the chains.

Let $x = \left\{\begin{array}{l}\text{the value of the gold in a braccio 30 links to an inch, in} \\ \text{francs,}\end{array}\right.$

$y =$ the value of the workmanship ....

Then $\frac{1}{30^2} : \frac{1}{90^2} = x : \frac{x}{9} \left\{\begin{array}{l}\text{the value of gold in a braccio 90 links to an} \\ \text{inch,}\end{array}\right.$

and $30 : 90 = y : 3y$ the value of the workmanship;

$$\therefore\ x + y = 3y, \text{ or } x = 2y \ldots\ldots\ldots (1),$$

$$\text{and } \frac{x}{9} + 3y = 3(x - y) + 4\tfrac{4}{9};$$

$$\therefore\ x + 27y = 27x - 27y + 40,$$

$$54y = 26x + 40,$$

$$27y = 13x + 20,$$

$$= 26y + 20, \text{ from (1)};$$

$\therefore\ y = 20$ francs, workmanship of 30 link-chain per braccio,

$x = 40$ .... gold ....

$\frac{x}{9} = 4\frac{4}{9}$ francs, gold in the other chain, per braccio,

$3y = 60$ .... workmanship ....

Prob. 16. A pack of np cards is dealt regularly round to p persons with their faces uppermost, every card dealt to each person being placed upon that previously dealt to him; the hands are then taken up, turned so as to have their backs uppermost, and placed upon one another; that hand which contains a particular card (A) being always placed below r other hands. The cards are then dealt again, the hands taken up, turned, and placed upon one another as before; and so on:—Shew that, if m and q be the whole numbers next greater than $\frac{\log 2np}{\log p}$ and $\frac{rnp}{p-1}$ respectively, the card A will, at the end of the m^{th} and every succeeding operation, occupy the q^{th} place, or be restricted to the and $\overline{q-1}$th places from the top, according as rn is indivisible or divi-le by $p-1$. (*Senate-House Prob.* 1839. *by Mr. Gaskin.*)

Let the 1st operation be performed, and let a be the number of rds which stand before A in its own hand; then $a<n$, and

$$rn + a = \text{N}^{\circ} \text{ of cards before } A \text{ in the pack,}$$

$$\left.\begin{array}{r}\text{whole N}^{\circ}\text{ next}\\ \text{less than...}\end{array}\right\} \quad \frac{rn+a}{p} = \left\{\begin{array}{l}\text{.............................. own hand,}\\ \text{after 2d operation,}\end{array}\right.$$

$$\text{..................} \quad rn + \frac{rn+a}{p} = \left\{\begin{array}{l}\text{.............................. the pack,}\\ \text{after 2d operation,}\end{array}\right.$$

$$\text{..................} \quad \frac{rn}{p} + \frac{rn+a}{p^2} = \left\{\begin{array}{l}\text{.............................. own hand,}\\ \text{after 3d operation,}\end{array}\right.$$

$$\text{.............} \quad rn + \frac{rn}{p} + \frac{rn+a}{p^2} = \left\{\begin{array}{l}\text{.............................. the pack,}\\ \text{after 3d operation,}\end{array}\right.$$

$$\text{\&c.} \qquad = \qquad \text{\&c.}$$

$\therefore$ whole number next less than $rn + \frac{rn}{p} + \frac{rn}{p^2} + \ldots + \frac{rn+a}{p^{m-1}}$,

that is, the whole number next less than $\frac{rn}{p^{m-1}} \cdot \frac{p^m - 1}{p-1} + \frac{a}{p^{m-1}}$,

r $\frac{rnp}{p-1} + \frac{a}{p^{m-1}} - \frac{rn}{p^{m-1}(p-1)} =$ number of cards before A in the pack, fter the mth operation.

Hence, if q be the whole number next greater than $\frac{rnp}{p-1}$, it is evi-ent, when $\frac{rn}{p-1}$ is not an integer, that the card A will be restricted to e qth place, provided $\frac{a}{p^{m-1}} - \frac{rn}{p^{m-1}(p-1)}$ is a fraction so small that it nnot make a difference of 1 in the value of

$$\frac{rnp}{p-1} + \frac{a}{p^{m-1}} - \frac{rn}{p^{m-1}(p-1)} \text{ ....(1).}$$

$$\text{Now } a<n, \quad \therefore \frac{a}{p^{m-1}} < \frac{np}{p^m}.$$

therefore $\frac{np}{p^m} =$ or $< \frac{1}{2}$, $\frac{a}{p^{m-1}} < \frac{1}{2}$, and $\frac{a}{p^{m-1}} - \frac{rn}{p^{m-1}(p-1)}$ is a small action,

$$\left(\text{for, since } r<p, \ \frac{rn}{p^{m-1}(p-1)} < \frac{1}{2}\right)$$

and $\therefore \frac{a}{p^{m-1}} - \frac{rn}{p^{m-1}(p-1)}$ cannot make a difference of 1 in the above expression (1).

Now $\frac{np}{p^m} = \frac{1}{2}$, or $p^m = 2np$, gives $m = \frac{\log 2np}{\log p}$.

Hence at the m^{th} and every succeeding operation the card A is restricted to the q^{th} place from the top.

If $\frac{rnp}{p-1}$ be an integer, that is, rn be divisible by $p-1$, then it is evident that the number of cards before A may be either $q-1$ or $q-2$; and therefore A must be restricted to the q^{th} or $\overline{q-1}^{\text{th}}$ place.

SINGLE AND DOUBLE POSITION.

Many problems are readily solved by means of the Arithmetical Rules called "Single Position," and "Double Position," without the application of Algebra; but the Rules themselves require algebraic proof.

(1) The Rule of 'Single Position' is applied to those cases only, in which the required quantity is some multiple, part, or parts, of some other *given* quantity; that is, if x represent the required value, a and b known quantities, the cases for 'Single Position' are such as produce an equation of the form

$$ax = b.$$

Thus, if it be required to find such a value x that $a \times x = b$, suppose s to be the value, and instead of b, we find $a \times s = b'$, then we have

$$\frac{ax}{as} = \frac{b}{b'},$$

$$\therefore\ x = \frac{b}{b'} \cdot s,$$

which points out the Rule:—namely, *Suppose some value* (s) *to be the one sought, and having operated upon it as the question directs, let the result* (b′) *be noted. Then the true value is equal to the true result* (b) *divided by the false one* (b′) *and multiplied by the supposed value* (s).

Ex. Find the number which being added to the half and fourth of itself will produce 14.

Suppose 12 the number, then

$$12 + \tfrac{1}{2} \text{ of } 12 + \tfrac{1}{4} \text{ of } 12 \text{ is } 21;$$

$$\therefore \text{ number required} = \frac{14}{21} \times 12 = 8.$$

(2) The rule for 'DOUBLE POSITION' is applied to those cases in which the required quantity is not a multiple, part, or parts, of a given quantity, but furnishes an equation of the form

$$ax + b = cx + d.$$

By transposition this equation becomes

$$(a - c)\,x + b - d = 0 \,\ldots\ldots\ldots\, (1).$$

Now suppose s to be the value of x, which by substitution does not satisfy the equation, but gives

$$(a - c)\,s + b - d = e \,\ldots\ldots\ldots\, (2),$$

then subtracting (1) from (2),

$$(a - c)\,(s - x) = e.$$

Again, suppose s' to be the value of x, which by substitution and subtraction, as before, gives

$$(a - c)\,(s' - x) = e'.$$

$$\text{Then } \frac{(a - c)\,(s - x)}{(a - c)\,(s' - x)} = \frac{e}{e'},$$

$$\text{or } \frac{s - x}{s' - x} = \frac{e}{e'},$$

$$\text{or } e's - e'x = es' - ex,$$

$$\therefore\ x = \frac{es' - e's}{e - e'},$$

which proves the common Rule, namely, *Make two suppositions* (s *and* s') *for the required quantity; treat each of them in the manner pointed out by the question; and note the errors* (e *and* e'); *then the required quantity will be found by dividing the difference of the products* es', e's *by the difference of the errors* e, e'.

Ex. What number is that which, upon being increased by 10, becomes three times as great as it was before?

1st. Suppose the number to be 20, (s)

$$\text{then } 20 + 10 = 30,$$

$$\text{but } 3 \times 20 = 60, \ \therefore\ e = -30.$$

2dly. Suppose the number to be (30), (s')

$$\text{then } 30 + 10 = 40,$$

$$\text{but } 3 \times 30 = 90, \ \therefore\ e' = -50.$$

$$\text{Hence the true number} = \frac{-30 \times 30 + 50 \times 20}{-30 + 50} = 5.$$

EXAMPLES.

VULGAR FRACTIONS.

1. **Reduce** to mixed numbers $\frac{602}{11}$, $\frac{4139}{15}$, $\frac{12332}{1111}$, $\frac{45739}{60}$.

Ans. $54\frac{8}{11}$, $275\frac{14}{15}$, $11\frac{111}{1111}$, $762\frac{19}{60}$.

2. What is $\frac{2}{5}$ of $9\frac{1}{6}$? also $\frac{5}{9}$ of $\frac{12}{25}$ of $\frac{1}{3}$ of $\frac{1}{2}$ of $41\frac{1}{5}$?

(1) Ans. $3\frac{2}{3}$. (2) Ans. $1\frac{187}{225}$.

3. Reduce to lowest terms $\frac{70}{462}$, $\frac{242}{1111}$, $\frac{799}{2961}$, $\frac{109375}{10000000}$.

Ans. $\frac{5}{33}$, $\frac{22}{101}$, $\frac{17}{63}$, $\frac{7}{640}$.

4. Reduce to the *least* common denominator $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$.

Ans. $\frac{210}{420}$, $\frac{140}{420}$, $\frac{105}{420}$, $\frac{84}{420}$, $\frac{70}{420}$, $\frac{60}{420}$.

5. Find the least common mult. of $2\frac{1}{2}$, $6\frac{1}{3}$, and 5. Ans. 95.

6. Reduce to the least common *numerator* $\frac{4}{13}$, $\frac{3}{10}$, $\frac{11}{36}$; and find which is the greatest. (1). Ans. $\frac{132}{429}$, $\frac{132}{440}$, $\frac{132}{432}$. (2). Ans. The 1st.

7. Which is the greatest $\frac{2}{3}$, $\frac{3}{4}$ or $\frac{4}{5}$? Ans. $\frac{4}{5}$.

8. Which is greater $7\frac{15}{16}$, or $7\frac{21}{23}$; and how much? Ans. $7\frac{15}{16}$ by $\frac{9}{368}$.

9. What is the difference between $7\frac{4}{5}$ and $7 \times \frac{4}{5}$? Ans. $2\frac{1}{5}$.

10. What fraction of £1 is 19*s.* $10\frac{3}{4}$*d.*? Ans. $\frac{191}{192}$.

11. What fraction of £5 is £3. 6*s.* 8*d*? Ans. $\frac{2}{3}$.

12. Which is the greater, $\sqrt{\frac{2}{3}}$, or $\sqrt[3]{\frac{2}{3}}$? Ans. The latter.

13. Reduce 5 yards 2 feet to the fraction of a mile. Ans. $\frac{17}{5280}$.

14. Which is greater, $\frac{1}{19}$ of £1, or $\frac{1}{20}$ of a guinea? Ans. $\frac{1}{19}$ of £1.

15. What is the difference between $\frac{1}{12}$ of £1, and $\frac{1}{14}$ of a guinea? Ans. 2*d.*

16. Add together $100\frac{5}{6}$, $1\frac{3}{4}$, $\frac{1}{3}$, and $7\frac{2}{5}$. Ans. $110\frac{19}{60}$.

17. Find the number which exceeds by $148\frac{2}{9}$ the difference between $5\frac{1}{3}$ and $95\frac{11}{12}$. Ans. $247\frac{23}{36}$.

18. Multiply $45\frac{3}{4}$ by $17\frac{2}{3}$, and divide the product by $4\frac{1}{4}$. Ans. $190\frac{3}{17}$.

19. Divide $2\frac{3}{4}$ of $7\frac{7}{11}$ by $\frac{1}{2}$ of $\frac{3}{4}$ of $18\frac{2}{3}$. Ans. 3.

20. Find the value of $7\frac{1}{3} \times \frac{9}{11} \times 17 \times \frac{1}{2} \times 8\frac{4}{9} \times \frac{1}{8}$. Ans. $53\frac{5}{6}$.

21. Reduce $\frac{12 \times 11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4 \times 5}$. Ans. 792.

22. Find the value of $\frac{3}{7} \times 1\frac{2}{5} \times 12\frac{1}{2} \div 6\frac{2}{3}$. Ans. $1\frac{1}{8}$.

23. Reduce $\frac{4\frac{1}{3} \times 4\frac{1}{3} \times 4\frac{1}{3} - 1}{4\frac{1}{3} \times 4\frac{1}{3} - 1}$ Ans. $4\frac{25}{48}$.

24. Reduce to simplest form $1\frac{1}{3} + \frac{8}{3}$ of $\frac{41}{34} + \frac{4}{5\frac{1}{10}}$. Ans. $5\frac{1}{3}$.

25. Reduce $\frac{5}{7} \times \frac{2}{9} \times 13\frac{1}{2} \div \left(\frac{1}{9} \times \frac{3}{7} + 54\right)$. Ans. $\frac{9}{227}$.

26. Reduce $\left(\frac{2}{19} + \frac{1}{3}\right) \div \left(3 - \frac{1}{3}\right) \times \left(\frac{1}{3} + \frac{1}{5}\right)$. Ans. $\frac{5}{57}$.

27. Reduce $\frac{18}{17} \times \left\{1 - \frac{64}{81}\right\} + \frac{8}{11} \times \frac{1}{6} \times \left(\frac{1}{2} + \frac{5}{12}\right)$. Ans. $\frac{1}{3}$.

28. Simplify the following peculiar fractional forms:

$\frac{\frac{1}{1}}{\frac{1}{3}}$, $\frac{\frac{2}{3}}{\frac{5}{11}}$, $\frac{1}{1+\frac{1}{2}}$, $2\frac{1}{2}+\frac{1}{3\frac{1}{3}+\frac{1}{4\frac{1}{4}}}$. Ans. 3, $1\frac{7}{15}$, $\frac{2}{3}$, $2\frac{71}{91}$.

29. Simplify $\frac{1}{2+\frac{1}{3+\frac{1}{4+\frac{1}{5}}}}$; and $\frac{2}{3+\frac{4}{5+\frac{6}{7}}}$.

(1) Ans. $\frac{68}{157}$.

(2) Ans. $\frac{82}{151}$.

30. $\frac{2}{3}$ multiplied by 3 signifies $\frac{2}{3}$ *taken three times*, that is, $\frac{2}{3}+\frac{2}{3}+\frac{2}{3}$; what does $\frac{2}{3}$ multiplied by $\frac{3}{4}$ signify?

31. If two-thirds of an estate be worth £220, what is the value of $\frac{3}{11}$ of the same? Ans. £90.

32. An article which cost 3*s*. 6*d*. is sold for 3*s*. 10$\frac{1}{2}$*d*.; what is that per cent. profit? Ans. $10\frac{5}{7}$.

33. How much per cent is 14*s*. 6*d*. of £3. 10*s*.? Ans. $20\frac{5}{7}$.

34. How much per cent. is $27\frac{1}{2}$ parts out of 36? Ans. $76\frac{7}{18}$.

35. A shilling weighs 3dwts. 15grs. of which 3 parts out of 40 are alloy, and the rest pure silver. How much per cent. is there of alloy, and what weight of pure silver?

(1) Ans. $7\frac{1}{2}$ per cent.; (2) Ans. 3dwts. $8\frac{19}{40}$ grs.

36. The length of $\frac{1}{360}$ of the Earth's circumference is $69\frac{1}{22}$ miles nearly; what is the Earth's diameter, assuming that the diameter of a circle is $\frac{7}{22}$ of its circumference? Ans. $7908\frac{102}{121}$ miles.

37. There are five numbers, of which the first two are $2\frac{1}{3}$, $3\frac{5}{24}$; and each number exceeds the preceding one by the same fraction; find the numbers, and the sum of them.

(1) Ans. $2\frac{1}{3}$, $3\frac{5}{24}$, $4\frac{1}{12}$, $4\frac{23}{24}$, $5\frac{5}{6}$. (2) Ans. $20\frac{5}{12}$.

38. What is the sum of which $\frac{7}{11}$ is 5*s*. 3*d*.? Ans. 8*s*. 3*d*.

39. Divide $\frac{3}{5}$ into two parts, so that one is greater than the other by $\frac{4}{13}$. Ans. $\frac{59}{130}$, $\frac{19}{130}$.

DECIMAL FRACTIONS.

1. Reduce to *vulgar* fractions 0·375, 0·8125, 4·075, 0·0064.

Ans. $\frac{3}{8}$, $\frac{13}{16}$, $4\frac{3}{40}$, $\frac{4}{625}$.

2. What is 0·003 of 0·27 of 90? Ans. 0·0729.

3. Divide 0·27 by 0·003; 0·06 by 60; 600 by 0·06; and 0·006 by 600.
(1) Ans. 90. (2) Ans. 0·001. (3) Ans. 10000. (4) Ans. 0·00001.

4. Reduce to their equivalent decimals $\frac{2}{25}$, $\frac{1}{800}$, $\frac{1}{128}$, $\frac{4000}{256}$.

Ans. 0·08, 0·00125, 0·0078125, 15·625.

5. Find the value in shillings and pence of £0·97216. Ans. 19*s.* $5\frac{1}{4}$*d.*

6. Reduce to vulgar fractions 0·8333...., 4·041666..., 0·09009009...

(1) Ans. $\frac{5}{6}$; (2) Ans. $4\frac{1}{24}$; (3) Ans. $\frac{10}{111}$.

7. Add together $2\frac{1}{3}$, $72\frac{5}{8}$, $316\frac{1}{6}$, and 2·875. Ans. 394.

8. Add together $1\frac{17}{51}$, $\frac{8}{3}$ of $\frac{41}{34}$, $\frac{4}{5\frac{1}{10}}$, and 0·6666..... Ans. 6.

9 What decimal of a square mile is one acre? Ans. 0·0015625.

10. What decimal of a year is 1 second? Ans. 0·0000000317.

11. Divide 0·454545.... by 0.121212...... Ans. 3·75.

12. Find the decimal which does not differ from $\frac{333}{106}$ by the ten-
iousandth part of an unit. Ans. 3·1415.

13. Shew that $3+\cfrac{1}{7+\cfrac{1}{16}} = 3\cdot14159$ nearly.

14. Divide $2\frac{1}{2}+\frac{1}{6}$ by $3\frac{1}{2}-\frac{1}{8}$; and express the result in a decimal
rm. Ans. 0·790123.

15. Extract the square root of $1\frac{3}{8}$ to 4 places of decimals; and the
ube-root of $3\frac{4}{5}$ to two places. (1) Ans. 1·1726. (2) Ans. 1·56.

16. Find $\frac{1}{\sqrt{3}}$ correct to 7 places of decimals; and $\sqrt{\frac{5\cdot04}{0\cdot012}}$ to two
laces. (1) Ans. 0·5773503. (2) Ans. 20·49.

17. Express in a decimal form $2+\frac{3}{5}+\frac{5}{1000}+\frac{14}{20000}$. Ans. 2·6057.

18. Reduce the following expressions to simple decimals having 7 decimal places:

$$\frac{1}{2} \times \{6\tfrac{1}{2} + 2\tfrac{2}{3} - 3\}.$$ Ans. 3·0833333.

$$2 \times \left\{\frac{1}{3} + \frac{1}{3} \times \frac{1}{3^3} + \frac{1}{5} \times \frac{1}{3^5} + \frac{1}{7} \times \frac{1}{3^7} + \dots\right\}.$$ Ans. 0·6931472.

$$2 \times \left\{\frac{1}{5} + \frac{1}{3} \times \frac{1}{5^3} + \frac{1}{5} \times \frac{1}{5^5} + \frac{1}{7} \times \frac{1}{5^7} + \&c.\right\}.$$ Ans. 0·4054650.

$$\frac{1}{2\left\{\frac{9}{11} + \frac{1}{3}\left(\frac{9}{11}\right)^3 + \frac{1}{5}\left(\frac{9}{11}\right)^5 + \&c.\right\}}.$$ Ans. 0·4342945.

19. Express the following in decimals of 4 places:

$$\frac{1}{2} + \frac{1}{3 \times 2^3} + \frac{1}{5 \times 2^5} + \frac{1}{7 \times 2^7}.$$ Ans. 0·5490.

$$16 \times \left\{\frac{1}{5} - \frac{1}{3 \times 5^3} + \frac{1}{5 \times 5^5} - \frac{1}{7 \times 5^7} + \&c.\right\} - \frac{4}{239}.$$ Ans. 3·1416.

20. Express a degree (69½ miles) in metres, 32 metres being equal to 35 yards nearly. Ans. 111835·42857.

21. The true length of a year is 365·24224 days. Find what the error amounts to by the common reckoning in 4 centuries.

Ans. 0·104 days.

22. A square inch plate of metal of 0·05 inches thickness is drawn into a wire of uniform thickness 50 feet long: find the thickness of the wire. Ans. Section of wire = 0·0000833 ... of a square inch.

23. A quadrant of the meridian in French metres is 10000565·278, and 1 metre = 39·37079 English inches: required the length of the quadrant in English feet. Ans. 32810846·2868.

24. The number of degrees in an arc of a circle which is equal to the radius is 57·29578: required the number of seconds in the same.

Ans. 206264·8.

25. Given that $\frac{1 \text{ French foot}}{1 \text{ English foot}} = 1{\cdot}0657654$, and that the equatoreal and polar radii of the Earth are respectively 3271953·854 and 3261072·9 *toises*, each *toise* being 6 French feet: find the Earth's equatoreal and polar radii in English feet. Ans. 20921665 and 20852394.

26. The length of the pendulum which vibrates seconds in the latitude of Greenwich is 39·1393 inches, and the force of gravity is measured by the product of (length of seconds pendulum) × $(3{\cdot}1415927)^2$: required the expression for the force of gravity in feet. Ans. 32·1908.

27. Two distances are measured in inches, and are known to be correct within a quarter of a hundredth of an inch each way, being 11·87 and 9·95. How far can their product be depended upon for accuracy?

Ans. Only in its integral part.

ALGEBRA.

NOTATION.

1. **Shew** that $\dfrac{x+1}{\dfrac{1}{x}+1}=x$, when $x=1$, or 2, or 3, or *any* number.

2. Shew that $\dfrac{7x+5}{23}+\dfrac{9x-1}{10}-\dfrac{x-9}{5}+\dfrac{2x-3}{15}=23\frac{1}{3}$, if $x=19$.

3. Find the value of $\dfrac{a+b}{a-b}$, when $a=\dfrac{1}{2}$ and $b=\dfrac{2}{5}$. Ans. 9.

4. Find the value of $\dfrac{x}{y}-\sqrt{\dfrac{1+x}{1-y}}$, when $x=\dfrac{1}{4}$ and $y=\dfrac{1}{5}$. Ans. 0.

5. Find the value of $5\sqrt{62+3x}-\dfrac{1}{2}\sqrt{95\frac{2}{3}-5x}$, when $x=6\frac{1}{3}$.
Ans. 41.

6. Find the value of $\sqrt{\dfrac{3}{4}-x}+\sqrt{2x}-\dfrac{3}{2}\sqrt{1-4x}$, when $x=\dfrac{1}{12}$.
Ans. 0.

7. Find the value of $\dfrac{3}{1+x}+\dfrac{3}{1-x}$, when $x=\dfrac{1}{2}$. Ans. 8.

8. Find $\dfrac{ax+by}{b+x}$, when $a=5$, $b=3$, $x=7$, $y=5$. Ans. 5.

9. Find $\dfrac{ax^2+b^2}{bx-a^2-c}$, when $a=3$, $b=5$, $c=2$, $x=6$. Ans. 7.

10. Find x^3-2xy^2+y-13, when $x=2$, and $y=-3$. Ans. -44.

11. Find the value of $\dfrac{x-1}{x+1}+\dfrac{x+3}{x-3}-2\,.\dfrac{x+2}{x-2}$, when $x=5$. Ans. 0.

12. Find the value of $\dfrac{a^2+b^2}{a^3-b^3}$, when $a=3$ and $b=-2$. Ans. $\dfrac{13}{35}$.

13. Find the value of $\dfrac{a+\sqrt{a^2+b^2}}{a^3-2b(a^2-b^2)}$, when $a=-4$, and $b=-3$.
Ans. $-\dfrac{1}{22}$.

14. Find the value of $3a^2b+\dfrac{a^2+b^2}{c}+abc+\dfrac{2ab}{3}+\dfrac{a^2}{b^3}+\dfrac{a-b}{a^3+b^3+c^4}$, when $a=4$, $b=3$, $c=2$. Ans. $189\frac{589}{5778}$.

ADDITION AND SUBTRACTION.

1. Add together $3x^2 - 5x + 1$, $7x^2 + 2x - 4$, and $-x^2 - 4x + 13$. Ans. $9x^2 - 7x + 10$.

2. Add together $a - 3b + 3c - d$, and $a + 3b + 3c + d$. Ans. $2a + 6c$.

3. Add together $a + b + c - d$, $a + b + d - c$, $a + c + d - b$, and $b + c + d - a$. Ans. $2(a + b + c + d)$.

4. Add together $x^3 - 2ax^2 + a^2x$, $x^3 + 3ax^2$, and $2a^3 - ax^2 - a^2x$. Ans. $2x^3 + 2a^3$.

5. Add together $a^m - b^n + 3x^p$, $2a^m - 3b^n - x^p$, and $a^m + 4b^n - x^q$. Ans. $4a^m + 2x^p - x^q$.

6. Add together $a^2 - 3ab - \frac{14}{21}b^2$, $2b^2 - \frac{2}{3}b^3 + c^2$, $ab - \frac{1}{3}b^2 + b^3$, and $2ab - \frac{1}{3}b^3$. Ans. $a^2 + b^2 + c^2$.

7. Add together $2(a + b) + c$, $2(a + b + c) + d$, and $2(a + b + c + d) + e$. Ans. $6(a + b) + 5c + 3d + e$.

8. Add together $(a + 7)x^2 - by^2$, and $(b - 7)x^2 - 7y^2$. Ans. $(a + b)x^2 - (b + 7)y^2$.

9. Add together $ax - by$, $x + y$, and $(a - 1)x - (b + 1)y$. Ans. $2ax - 2by$.

10. Add together $2(a + b)xy^2$, $3(a - 2b)xy^2$, and $4(2a - b)xy^2$. Ans. $(13a - 8b)xy^2$.

11. Add together $at^n + 3bt^n - 9at^n + 7bt^n$. Ans. $(10b - 8a)t^n$.

12. Add together $1 - (1 - \overline{1 - x})$, $2x - (3 - 5x)$, and $2 - (-4 + 5x)$. Ans. $4 + x$.

13. From $6a - b - c$ take $a - b + 2c$. Ans. $5a - 3c$.

14. From $8a + x - 5b - 5c$ take $x + 2b - 5c$. Ans. $8a - 7b$.

15. From $a + b$ take $\frac{a}{2} - \frac{b}{2}$. Ans. $\frac{a}{2} + \frac{3b}{2}$.

16. From $\frac{a+b}{2}$ take $\frac{a-b}{2}$; and $\frac{1}{2}(a + b)$ from $a - b$. (1) Ans. b, (2) Ans. $\frac{a}{2} - \frac{3b}{2}$.

17. From $2(a + b) - 3(c - d)$ take $a + b - 4(c - d)$. Ans. $a + b + c - d$.

18. From $(a + b)x + (b + c)y$ take $(a - b)x - (b - c)y$. Ans. $2b(x + y)$.

19. From $(a^2+bc)x^2-(a^2-c^2)bx$ take $bcx^2-(a^2-b^2)bx$.
Ans. $a^2x^2-(b^2-c^2)bx$.

20. From x^3-ax^2+bx-c take x^3-px^2+qx-r.
Ans. $(p-a)x^2-(q-b)x+r-c$.

21. From $a-x-(x-2a)+2a-x$ take $a-2x-(2a-x)+(x-2a)$.
Ans. $8a-3x$.

22. Simplify the following quantities:

$a-\{b-(2b+x)\}+\{b-(x-2b)\}$. Ans. $a+4b$.

$a-(b-c)-(a-c)+c-(a-b)$. Ans. $3c-a$.

$a-\{a+b-[a+b+c-(a+b+c+d)]\}$. Ans. $-b-d$.

$a+b-(2a-3b)-(5a+7b)-(-13a+2b)$. Ans. $7a-5b$.

MULTIPLICATION.

1. Multiply $4a^2-3ac+2$ by $5ax$. Ans. $20a^3x-15a^2cx+10ax$.

2. Multiply $5a-2ab+10$ by $-9ab$. Ans. $-45a^2b+18a^2b^2-90ab$.

3. Multiply $2x+3y$ by $2x-3y$. Ans. $4x^2-9y^2$.

4. Multiply $4a^2-6a+9$ by $2a+3$. Ans. $8a^3+27$.

5. Multiply $a^4+a^3b+a^2b^2+ab^3+b^4$ by $a-b$. Ans. a^5-b^5.

6. Multiply $2a+bc-2b^2$ by $2a-bc+2b^2$.
Ans. $4a^2-b^2c^2+4b^3c-4b^4$.

7. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$.
Ans. $a^6-3a^4b^2+3a^2b^4-b^6$.

8. Multiply $4ab-2ac$ by $6ab+3ac$. Ans. $6a^2(4b^2-c^2)$.

9. Find the product of $a-b+c-d$, and $a+b-c-d$.
Ans. $a^2-b^2-c^2+d^2-2ad+2bc$.

10. Multiply $a+bx$ by $a+cx$. Ans. $a^2+(ab+ac)x+bcx^2$.

11. Multiply x^3-ax^2+bx-c by x^2-px+q.
Ans. $x^5-(a+p)x^4+(b+ap+q)x^3-(c+pb+aq)x^2+(bq+cp)x-qc$.

12. Find the product of $(x-a)(x-b)(x-c)$.
Ans. $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$.

13. Find the product of $(x-10)(x+1)(x+4)$. Ans. $x^3-5x^2-46x-40$.

14. Find the product of $(x-5)(x+6)(x-7)(x+8)$.
Ans. $x^4+2x^3-85x^2-86x+1680$.

15. Find the continued product of $x+1$, $x+2$, $x+3$, and $x+4$.
Ans. $x^4+10x^3+35x^2+50x+24$.

16. Find the product of a^2+ax+x^2 and a^2-ax+x^2. Ans. $a^4+a^2x^2+x^4$.

17. Find the continued product of $x-a$, $x+a$, x^2-ax+a^2, and x^2+ax+a^2.
Ans. x^6-a^6.

18. Multiply $x^2-\frac{1}{2}x+\frac{2}{3}$ by $\frac{1}{3}x+2$. Ans. $\frac{1}{3}x^3+\frac{11}{6}x^2+\frac{7}{9}x+\frac{4}{3}$.

19. Multiply x^3-x^{-3} by $x-x^{-1}$. Ans. $x^4+x^{-4}-x^2-x^{-2}$.

20. Multiply $(1+a)a^2y+y^2+ay^2$ by a^2-y. Ans. $(1+a)y(a^4-y^2)$.

21. Multiply $a^{\frac{4}{5}}+a^{\frac{3}{5}}x^{\frac{1}{5}}+a^{\frac{2}{5}}x^{\frac{2}{5}}+a^{\frac{1}{5}}x^{\frac{3}{5}}+x^{\frac{4}{5}}$ by $a^{\frac{1}{5}}-x^{\frac{1}{5}}$. Ans. $a-x$.

22. Multiply $x+2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$ by $x-2y^{\frac{1}{2}}+3z^{\frac{1}{3}}$.
Ans. $x^2-4y+6xz^{\frac{1}{3}}+9z^{\frac{2}{3}}$.

23. Multiply $a^{\frac{5}{2}}-2a^2b^{\frac{1}{3}}+4a^{\frac{3}{2}}b^{\frac{2}{3}}-8ab+16a^{\frac{1}{2}}b^{\frac{4}{3}}-32b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}+2b^{\frac{1}{3}}$.
Ans. a^3-64b^2.

24. Multiply $3a^{\frac{1}{2}}b^{\frac{2}{3}}+5a^{\frac{1}{3}}b^{\frac{2}{3}}$ by $3a^{\frac{1}{2}}b^{\frac{2}{3}}-5a^{\frac{1}{3}}b^{\frac{2}{3}}$. Ans. $9ab^{\frac{4}{3}}-25a^{\frac{2}{3}}b^{\frac{4}{3}}$.

25. Multiply $a^{2-2m}b^{2p+1}c^3$ by $a^{5m-2}b^{-2}c^{-1}$. Ans. $a^{3m}b^{2p-1}c^2$.

26. Multiply a^m-2c^n by a^m-c^n. Ans. $a^{2m}-3a^mc^n+2c^{2n}$.

27. Multiply $x^{-3p}+ax^{-2p}-a^2x^{-p}$ by a^mx^p. Ans. $a^mx^{-2p}+a^{m+1}x^{-p}-a^{m+2}$.

28. Required the coefficient of x^4 in the product of $x^4-ax^3+bx^2-cx+d$ and x^2+px+q. Ans. $b-ap+q$.

29. Multiply $(2b-c)a^2-(4b^2-2bc+c^2)a+8b^3-4b^2c$ by $(2b+c)a$.
Ans. $(4b^2-c^2)a^3-(8b^3+c^3)a^2+(16b^4-4b^2c^2)a$.

30. Multiply $a^{(p-1)q}-b^{(q-1)p}$ by a^q-b^p. Ans. $a^{pq}-a^qb^{(q-1)p}-b^pa^{(p-1)q}+b^{pq}$.

31. Multiply $1+\frac{1}{2}x+\frac{1}{3}x^2+\frac{1}{4}x^3+\ldots\ldots$ by $1-\frac{1}{3}x+\frac{1}{5}x^2-\frac{1}{7}x^3+\ldots$
Ans. $1+\frac{1}{6}x+\frac{11}{30}x^2+\frac{121}{1260}x^3+\ldots$

32. Simplify $\frac{1}{6}\{x(x+1)(x+2)+x(x-1)(x-2)\}+\frac{2}{3}(x-1)x(x+1)$.
Ans. x^3.

33. Find the difference between $a(b+c)^2+b(a+c)^2+c(a+b)^2$ and $(a+b)(a-c)(b-c)+(a-b)(a-c)(b+c)-(a-b)(b-c)(a+c)$.
Ans. $12abc$.

DIVISION.

1. DIVIDE $5xy + 20x^2y - 45axy$ by $5xy$. Ans. $1 + 4x - 9a$.

2. Divide $-9a^2bc - 12ab^2c + 15abc^2$ by $-3abc$. Ans. $3a + 4b - 5c$.

3. Divide $2a^2 + a - 6$ by $2a - 3$. Ans. $a + 2$.

4. Divide $a^5 + b^5$ by $a + b$. Ans. $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.

5. Divide $x^6 - 2a^3x^3 + a^6$ by $x^2 - 2ax + a^2$.
Ans. $x^4 + 2ax^3 + 3a^2x^2 + 2a^3x + a^4$.

6. Divide $x^6 - a^6$ by $x^3 + 2ax^2 + 2a^2x + a^3$. Ans. $x^3 - 2ax^2 + 2a^2x - a^3$.

7. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
Ans. $a^2 + b^2 + c^2 - ab - ac - bc$.

8. Divide $m^2 + 2mp - n^2 - 2nq + p^2 - q^2$ by $m - n + p - q$.
Ans. $m + n + p + q$.

9. Divide $1 + 2x$ by $1 - 3x$. Ans. $1 + 5x + 15x^2 + 45x^3 + \ldots$

10. Divide $a^4 - 81$ by $a - 3$. Ans. $a^3 + 3a^2 + 9a + 27$.

11. Divide $552a^{11}b^3x^4y^3z$ by $184a^6b^3x^2yz$. Ans. $3a^5x^2y^2$.

12. Divide $115a^mb^nc^{2p+q}d^{2r-1}$ by $-69a^nb^nc^{p+q}d$. Ans. $-\frac{5}{3}a^{m-n}c^pd^{2r-2}$.

13. Divide $a + b$ by $\sqrt[3]{a} + \sqrt[3]{b}$. Ans. $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$.

14. Divide $x^{m+n} + x^ny^n + x^my^m + y^{m+n}$ by $x^n + y^m$. Ans. $x^m + y^n$.

15. Divide $ax^3 - (a^2 + b)x^2 + b^2$ by $ax - b$. Ans. $x^2 - ax - b$.

16. Divide $x^3 - apx^2 + a^2px - a^3$ by $x - a$. Ans. $x^2 - apx + ax + a^2$.

17. Divide $x^5 - px^4 + qx^3 - qx^2 + px - 1$ by $x - 1$.
Ans. $x^4 - (p - 1)x^3 + (q - p + 1)x^2 - (p - 1)x + 1$.

18. Divide $mpx^3 + (mq - np)x^2 - (mr + nq)x + nr$ by $mx - n$.
Ans. $px^2 + qx - r$.

19. Divide $(x^3 - 1)a^3 - (x^3 + x^2 - 2)a^2 + (4x^2 + 3x + 2)a - 3(x + 1)$ by $(x - 1)a^2 - (x - 1)a + 3$. Ans. $(x^2 + x + 1)a - (x + 1)$.

20. Divide $x(x - 1)a^3 + (x^3 + 2x - 2)a^2 + (3x^2 - x^3)a - x^4$ by $a^2x + 2a - x^2$.
Ans. $(x - 1)a + x^2$.

21. Divide $-2a^{-8}x^5 + 17a^{-4}x^6 - 5x^7 - 24a^4x^8$ by $2a^{-3}x^3 - 3ax^4$.
Ans. $-a^{-5}x^2 + 7a^{-1}x^3 + 8a^3x^4$.

22. Divide $(2x-y)^2a^4-(x+y)^2a^2x^2+(x+y)2ax^4-x^6$ by $(2x-y)a^2-(x+y)ax+x^3$. Ans. $(2x-y)a^2+(x+y)ax-x^3$.

23. Divide $(3c-6b)a^2-(c^2-4b^2)a+c^3-6bc^2+12b^2c-8b^3$ by $c-2b$. Ans. $3a^2-(c+2b)a+c^2-4bc+4b^2$.

24. Divide $\frac{3x^5}{4}-4x^4+\frac{77x^3}{8}-\frac{43x^2}{4}-\frac{33x}{4}+27$ by $\frac{x^2}{2}-x+3$.

Ans. $\frac{3x^3}{2}-5x^2+\frac{x}{4}+9$.

25. Divide $\frac{x^4}{3}-\frac{11x^3}{12}+\frac{41x^2}{8}-\frac{23x}{4}+6$ by $\frac{2x^2}{3}-\frac{5x}{6}+1$.

Ans. $\frac{x^2}{2}-\frac{3x}{4}+6$.

26. Divide $\frac{2}{5}a^4x-\frac{136}{75}a^3x^2+\frac{8}{5}a^2x^3+\frac{3}{10}ax^4-x^5$ by $\frac{2}{3}a^3-\frac{4}{5}a^2x+\frac{1}{2}x^2$.

Ans. $\frac{3}{5}ax-2x^3$

27. Divide $\frac{1}{2}a^{\frac{3}{4}}-\frac{3}{10}a^{\frac{7}{10}}-\frac{1}{3}a^{\frac{7}{12}}+\frac{1}{5}a^{\frac{8}{15}}+\frac{1}{4}a^{\frac{1}{2}}-\frac{3}{20}a^{\frac{9}{20}}$ by $\sqrt[4]{a}-\frac{3}{5}\sqrt[5]{a}$.

Ans. $\frac{1}{2}\sqrt{a}-\frac{1}{3}\sqrt[3]{a}+\frac{1}{4}\sqrt[4]{a}$.

28. Divide $x^{\frac{3}{2}}-y^{\frac{3}{2}}$ by $x^{\frac{1}{2}}-y^{\frac{1}{2}}$. Ans. $x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y$.

29. Divide $a-b$ by $\sqrt{a}-\sqrt[4]{b}$. Ans. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}$.

30. Divide $a^{3m-2n}b^{2p}c-a^{2m+n-1}b^{1-p}c^n+a^{-n}b^{-1}c^m+a^{3m-n}b^{3p+2}c^n-a^{2m+2n-1}b^3c^{2n-1}$ $+b^{p+1}c^{m+n-1}$ by $a^{-n}b^{-p-1}+bc^{n-1}$. Ans. $a^{3m-n}b^{3p+1}c-a^{2m+2n-1}b^2c^n+b^pc^m$.

31. Divide $x^{pq}-1$ by x^q-1, and write down the last three terms of the quotient. Ans. $x^{q(p-1)}+x^{q(p-2)}+x^{q(p-3)}+\ldots x^{2q}+x^q+1\ldots$

N.B. Since the Product of two quantities ÷ Multiplicand = Multiplier, or Product ÷ Multiplier = Multiplicand, each Example in Multiplication supplies *two others* in Division. Also, since Dividend ÷ Quotient = Divisor, or Divisor × Quotient = Dividend, each Example in Division supplies *two others*, one in Division, the other in Multiplication.

GREATEST COMMON MEASURE.

1. FIND the Greatest Common Measure of $3a^4x^5y$ and $6a^2bx$. Ans. $3a^2x$.

2. Find the G. C. M of $ax+x^2$ and $abc+bcx$. Ans. $a+x$.

3. Find the G. C. M. of $a^2+ab-12b^2$ and $a^2-5ab+6b^2$. Ans. $a-3b$.

4. Find the G. C. M. of $6a^2+7ax-3x^2$ and $6a^2+11ax+3x^2$.
Ans. $2a+3x$.

5. Find the G. C. M. of $x^4+a^2x^2+a^4$ and $x^4+ax^3-a^3x-a^4$.
Ans. x^2+ax+a^2.

6. Find the G. C. M. of $20x^4+x^2-1$ and $25x^4+5x^3-x-1$.
Ans. $5x^2-1$.

7. Find the G. C. M. of $6a^3-6a^2y+2ay^2-2y^3$ and $12a^2-15ay+3y^2$.
Ans. $a-y$.

8. Find the G. C. M. of $48x^2+16x-15$ and $24x^3-22x^2+17x-5$.
Ans. $12x-5$.

9. Find the G. C. M. of $6x^5-4x^4-11x^3-3x^2-3x-1$ and $4x^4+2x^3-18x^2+3x-5$.
Ans. $2x^3-4x^2+x-1$.

10. Find the G. C. M. of $x^4+ax^3-9a^2x^2+11a^3x-4a^4$ and $x^4-ax^3-3a^2x^2+5a^3x-2a^4$.
Ans. $(x-a)^3$.

11. Find the G. C. M. of $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$.
Ans. x^2-1.

12. Find the G. C. M. of $3x^2-(4a+2b)x+2ab+a^2$ and $x^3-(2a+b)x^2+(2ab+a^2)x-a^2b$.
Ans. $x-a$.

13. Find the G. C. M. of $2x^3+(2a+3b)x^2+(2b+3ab)x+3b^2$ and $2x^2+(2c+3b)x+3bc$.
Ans. $2x+3b$.

14. Find the G. C. M. of x^2+5x+4, x^2+2x-8, and $x^2+7x+12$.
Ans. $x+4$.

15. Find the G. C. M. of $15a^4+10a^3b+4a^2b^2+6ab^3-3b^4$ and $6a^3+19a^2b+8ab^2-5b^3$.
Ans. $3a^2+2ab-b^2$.

16. Find the G. C. M. of $ab+2a^2-3b^2-4bc-ac-c^2$ and $9ac+2a^2-5ab+4c^2+8bc-12b^2$.
Ans. $2a+3b+c$.

17. Find the G. C. M. of $qnp^3+3np^2q^2-2npq^3-2nq^4$ and $2mp^2q^2-4mp^4-mp^3q+3mpq^3$.
Ans. $p-q$.

18. Find the G. C. M. of $x^6+4x^5-3x^4-16x^3+11x^2+12x-9$ and $6x^5+20x^4-12x^3-48x^2+22x+12$.
Ans. x^3+x^2-5x+3.

19. Find the G. C. M. of $a^2+2b^2+\sqrt{ab}\,(a+2b)$ and $a^2-b^2+(a-b)\sqrt{ab}$.
Ans. $\sqrt{a}+\sqrt{b}$.

20. Find the G. C. M. of $\frac{x^2}{3}+\frac{11x}{6}\sqrt{x+1}-x-1$ and $x^2-\frac{x}{4}-\frac{1}{4}$.
Ans. $x-\frac{1}{2}\sqrt{x+1}$.

21. Find the G. C. M. of $a^2x^3 + a^5 - 2abx^3 + b^2x^3 + a^3b^2 - 2a^4b$ and $2a^2x^4 - 5a^4x^2 + 3a^6 - 2b^2x^4 + 5a^2b^2x^2 - 3a^4b^2$. Ans. $(a-b)(x+a)$.

22. Find the G. C. M. of
$2(y^3-2y^2-y+2)x^3 + 3(y^2-1)x^2 - (2y^3-y^2-2y+1)$ and
$3(y^3-4y^2+5y-2)x^2 + 7(y^2-2y+1)x - (3y^3-5y^2+y+1)$. Ans. $xy-x-y+1$.

23. Find the G. C. M. of $(b-c)x^2 + (2ab-2ac)x + a^2b - a^2c$ and $(ab-ac+b^2-bc)x + a^2c + ab^2 - a^2b - abc$. Ans. $b-c$.

24. Find the value of y which will make
$2(y^2+y)x^2 + (11y-2)x + 4$ and $2(y^3+y^2)x^3 + (11y^2-2y)x^2 + (y^2+5y)x + 5y - 1$ have a common measure. Ans. $y=5$.

25. Find the G. C. M. of $6ax^2 - 6ab^2 - 54a - 21x^2 + 21b^2 + 189$, $6ax^5 + 60a - 21x^5 - 210$, and $6ab^3 - 12a - 21b^3 + 42$. Ans. $6a-21$.

26. Find the G. C. M. of $6a^3x^2 + 4ax^3 - 10axy - 3a^2xy - 2x^2y + 5y^2$, $10a^2x^2 - 2ax^3y - 6axy^2 - 5axy + x^2y^2 + 3y^3$, and $-4ax^2 - 6a^2bxy + 2ab^2x^2y - 2abx + 2xy + 3aby^2 - b^2xy^2 + by$. Ans. $y-2ax$.

N. B. Since each proposed quantity is divisible by the G.C.M. each Example in Greatest Common Measure supplies two or more Examples in Division.

LEAST COMMON MULTIPLE.

1. FIND the Least Common Multiple of $8a^2$, $12a^3$, and $20a^4$. Ans. $120a^4$.

2. Find the L. C. M. of $1-a$, $1+a$, and $1-a^2$. Ans. $1-a^2$.

3. Find the L. C. M. of a^3-x^3 and a^2-x^2. Ans. $a^4+a^3x-ax^3-x^4$.

4. Find the L. C. M. of $2x-1$, $4x^2-1$, and $4x^2+1$. Ans. $16x^4-1$.

5. Find the L. C. M. of x^2+5x+4, x^2+2x-8, and $x^2+7x+12$. Ans. $x^4+6x^3+3x^2-26x-24$.

6. Find the L. C. M. of a^2-b^2, a^2+b^2, $(a-b)^2$, $(a+b)^2$, a^3-b^3 and a^3+b^3. Ans. $a^{10}-a^6b^4-a^4b^6+b^{10}$.

7. Find the L. C. M. of $x-1$, x^2-1, $x-2$, and x^2-4. Ans. x^4-5x^2+4.

8. Find the L. C. M. of $4(1-x)^2$, $8(1-x)$, $8(1+x)$, and $4(1+x^2)$. Ans. $8(1-x)(1-x^4)$.

9. Find the L. C. M. of $3x^2-11x+6$, $2x^2-7x+3$, and $6x^2-7x+2$. Ans. $6x^3-25x^2+23x-6$.

10. Find the L. C. M. of x^3-3x^2+3x-1, x^3-x^2-x+1, x^4-2x^3+2x-1, and $x^4-2x^3+2x^2-2x+1$. Ans. $x^6-2x^5+x^4-x^2+2x-1$.

11. Shew that if $x+c$ be the G. C. M. of x^2+ax+b, and $x^2+a'x+b'$, their least com. mult. will be $x^3+(a+a'-c)x^2+(aa'-c^2)x+(a-c)(a'-c)c$.

FRACTIONS.

Reduce the following fractional expressions:—

1. $\frac{3x-1}{24}-\frac{3x-5}{24}+\frac{5}{6}$. Ans. 1.

2. $\frac{4m-3n}{3(1-n)}-\frac{m+3n}{3(1-n)}+\frac{2n}{1-n}$. Ans. $\frac{m}{1-n}$.

3. $\frac{a-b}{ab}-\frac{a-c}{ac}+\frac{b-c}{bc}$. Ans. 0.

4. $\frac{7x-10}{5}-\frac{3x-7}{6}-\frac{27x-30}{30}$. Ans. $\frac{1}{6}$.

5. $\frac{3x-4y}{7}-\frac{2x-y-1}{3}+\frac{15x-4}{12}$. Ans. $\frac{85x-20y}{84}$.

6. $\frac{a}{c}-\frac{(ad-bc)x}{c(c+dx)}-\frac{a}{c+dx}$. Ans. $\frac{bx}{c+dx}$.

7. $\dfrac{a+\frac{b-a}{1+ba}}{1-a.\frac{b-a}{1+ba}}$; $2.\dfrac{x^2-\frac{1}{4}}{2x+1}+\frac{1}{2}$. (1) Ans. b. (2) Ans. x.

8. $\dfrac{\frac{1}{a}+\frac{1}{ab^3}}{b-1+\frac{1}{b}}$; $\dfrac{a+b+\frac{b^2}{a}}{a+b+\frac{a^2}{b}}$. (1) Ans. $\frac{b+1}{ab^2}$. (2) Ans. $\frac{b}{a}$.

9. $\frac{1}{n-1-(n-1)x}-\frac{1}{n+1+(n+1)x}$. Ans. $\frac{2}{n^2-1}.\frac{1+nx}{1-x^2}$.

10. $\frac{a}{b}-\frac{(a^2-b^2)x}{b^2}+\frac{a(a^2-b^2)x^2}{b^2(b+ax)}$. Ans. $\frac{a+bx}{b+ax}$.

11. $\frac{1}{2}.\frac{3m+2n}{3m-2n}-\frac{1}{2}.\frac{3m-2n}{3m+2n}$. Ans. $\frac{6mn}{9m^2-4n^2}$.

12. $\frac{x+y}{y}-\frac{2x}{x+y}+\frac{x^3-x^2y}{y^3-x^2y}$. Ans. $\frac{y}{x+y}$.

13. $\frac{x^4-y^4}{x^2-2xy+y^2}\div\frac{x^2+xy}{x-y}$. Ans. $x+\frac{y^2}{x}$.

14. $\frac{x(x+1)(x+2)}{3}-\frac{x(x+1)(2x+1)}{1\,.\,2\,.\,3}$. Ans. $\frac{x(x+1)}{2}$.

15. $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}$. Ans. $\frac{x+3}{x^4-1}$.

16. $\frac{3}{4(1-x)^2}+\frac{3}{8(1-x)}+\frac{1}{8(1+x)}-\frac{1-x}{4(1+x^2)}$. Ans. $\frac{1+x-x^2}{1-x-x^4+x^5}$.

17. $\frac{3a}{(a-2x)^2}+\frac{2a+x}{(a+x)(a-2x)}-\frac{5}{a+x}$. Ans. $\frac{20ax-22x^2}{(a+x)(a-2x)^2}$.

18. $\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}$. Ans. $\frac{1}{a^4-x^4}$.

19. $\frac{x^{3n}}{x^n-1}-\frac{x^{2n}}{x^n+1}-\frac{1}{x^n-1}+\frac{1}{x^n+1}$. Ans. $x^{2n}+2$.

20. $\frac{a^2+b^2+ab}{(a^3-b^3)(x^{\frac{1}{2}}-a^{\frac{1}{2}})}-\frac{x^{\frac{1}{2}}}{(a-b)(x-a)}$. Ans. $\frac{a^{\frac{1}{2}}}{(a-b)(x-a)}$.

21. $\frac{8}{5(x-2)}-\frac{13}{80(x+3)}-\frac{5}{4(x-1)^2}-\frac{23}{16(x-1)}$.

Ans. $\frac{x^2+4}{(x-1)^2(x-2)(x+3)}$.

22. $\frac{1}{(a-b)(a-c)(x+a)}-\frac{1}{(a-b)(b-c)(x+b)}+\frac{1}{(a-c)(b-c)(x+c)}$.

Ans. $\frac{1}{(x+a)(x+b)(x+c)}$.

23. $\frac{a+b}{ab}(a^2+b^2-c^2)+\frac{b+c}{bc}(b^2+c^2-a^2)+\frac{a+c}{ac}(a^2+c^2-b^2)$.

Ans. $2(a+b+c)$.

24. $\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(c-a)^2}{(a+b)^2-c^2}+\frac{c^2-(a-b)^2}{(b+c)^2-a^2}$. Ans. 1.

25. $\dfrac{1}{a+\dfrac{1}{b+\dfrac{1}{c}}}$, $\dfrac{1}{x-1+\dfrac{1}{1+\dfrac{x}{4-x}}}$. (1) Ans. $\frac{bc+1}{abc+a+c}$.

(2) Ans. $\frac{4}{3x}$.

26. Reduce to lowest terms $\frac{ax}{3ax}$, and $\frac{m^2-n^2}{3m-3n}$. Ans. $\frac{1}{3}$, $\frac{m+n}{3}$.

27. Reduce to lowest terms $\frac{2a^3-2ab}{5a^3-5ab}$, $\frac{ax+x^2}{3bx-cx}$, $\frac{14a^2-7ab}{10ac-5bc}$, $\frac{12a^3x^4+2a^2x^5}{18ab^2x+3b^2x^2}$, and $\frac{5a^2+5ax}{a^2-x^2}$. Ans. $\frac{2}{5}$, $\frac{a+x}{3b-c}$, $\frac{7a}{5c}$, $\frac{2a^2x^3}{3b^2}$, $\frac{5a}{a-x}$.

28. Reduce to lowest terms $\frac{x^2+(a+c)x+ac}{x^2+(b+c)x+bc}$. Ans. $\frac{x+a}{x+b}$.

29. Reduce to lowest terms $\frac{ac+by+ay+bc}{af+2bx+2ax+bf}$. Ans. $\frac{c+y}{f+2x}$.

30. Reduce to lowest terms the following fractions:—

$$\frac{6ac+10bc+9ax+15bx}{6c^2+9cx-2c-3x}. \qquad \text{Ans. } \frac{3a+5b}{3c-1}.$$

31. $$\frac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}. \qquad \text{Ans. } \frac{a+b+c}{a-b-c}.$$

32. $$\frac{x^8+x^6y^2+x^2y+y^3}{x^4-y^4}. \qquad \text{Ans. } \frac{x^6+y}{x^2-y^2}.$$

33. $$\frac{a^3+ab^2-a^2b-b^3}{4a^4-2a^2b^2-4a^3b+2ab^3}. \qquad \text{Ans. } \frac{a^2+b^2}{2a(2a-b^2)}.$$

34. $$\frac{2a^2+ab-b^2}{a^3+a^2b-a-b}. \qquad \text{Ans. } \frac{2a-b}{a^2-1}.$$

35. $$\frac{6x^5+15x^4y-4x^3z^2+10x^2yz^2}{9x^3y-27x^2yz-6xyz^2+18yz^3}. \qquad \text{Ans. } \frac{x^2}{3y}\cdot\frac{2x+5y}{x-3z}.$$

36. $$\frac{acx^2+(ad+bc)x+bd}{a^2x^2-b^2}. \qquad \text{Ans. } \frac{cx+d}{ax-b}.$$

37. $$\frac{a^3+(a+b)ax+bx^2}{a^4-b^2x^2}. \qquad \text{Ans. } \frac{a+x}{a^2-bx}.$$

38. $$\frac{4x^2-12ax+9a^2}{8x^3-27a^3}. \qquad \text{Ans. } \frac{2x-3a}{4x^2+6ax+9a^2}.$$

39. $$\frac{ax^m-bx^{m+1}}{a^2bx-b^3x^3}. \qquad \text{Ans. } \frac{x^{m-1}}{b(a+bx)}.$$

40. $$\frac{30a^{3n-1}b^rc^{r+2}-6a^{2n-4}b^3c^rd^{r-1}}{20a^nb^{r-1}c^2d^2-4a^{-3}b^2d^{r+1}}. \qquad \text{Ans. } \frac{3a^{2n-1}bc^r}{2d^2}.$$

41. $$\frac{a^2(b^2-c^2)-ab(2b^2+bc-c^2)+b^3(b+c)}{a^3(b^2+2bc+c^2)-a^2b(2b^2+3bc+c^2)+ab^3(b+c)}. \qquad \text{Ans. } \frac{a(b-c)-b^2}{a^2(b+c)-ab^2}.$$

42. $$\frac{(c-d)a^2+6(bc-bd)a+9(b^2c-b^2d)}{(bc-bd+c^2-cd)a+3(b^2c+bc^2-b^2d-bcd)}. \qquad \text{Ans. } \frac{a+3b}{b+c}.$$

43. $$\frac{1+x^{\frac{1}{2}}+x+x^{\frac{3}{2}}}{2x+2x^{\frac{3}{2}}+3x^2+3x^{\frac{5}{2}}}. \qquad \text{Ans. } \frac{1+x}{2x+3x^2}.$$

44. $$\frac{3bcq+30mp+18bc+5mpq}{4adq-42fg+24ad-7fgq}. \qquad \text{Ans. } \frac{3bc+5mp}{4ad-7fg}.$$

45. $$\frac{x-4-3x^{\frac{1}{2}}+4y^{\frac{1}{2}}-(xy)^{\frac{1}{2}}}{x-8-2x^{\frac{1}{2}}+12y^{\frac{1}{2}}-3(xy)^{\frac{1}{2}}}. \qquad \text{Ans. } \frac{x^{\frac{1}{2}}-y^{\frac{1}{2}}+1}{x^{\frac{1}{2}}-3y^{\frac{1}{2}}+2}.$$

1. MULTIPLY $a-\frac{x^2}{a}$ by $\frac{a}{x}+\frac{x}{a}$. Ans. $\frac{a^2}{x}-\frac{x^3}{a^2}$.

2. Multiply $a+\frac{1}{a}$ by $a+\frac{1}{a}$. Ans. $a^2+\frac{1}{a^2}+2$.

3. Multiply $\frac{b}{a}+b$ by $\frac{a}{b}+\frac{1}{b}$. Ans $a+\frac{1}{a}+2$.

4. Multiply $\frac{2a-b}{4a}$ by $\frac{6a-2b}{b^2-2ab}$. Ans. $\frac{b-3a}{2ab}$.

5. Multiply x^2+x+1 by $\frac{1}{x^2}-\frac{1}{x}+1$. Ans. $x^2+1+\frac{1}{x^2}$.

6. Multiply $x+1+\frac{1}{x}$ by $x-1+\frac{1}{x}$. Ans. $x^2+1+\frac{1}{x^2}$.

7. Multiply $\frac{a}{a+b}+\frac{b}{a-b}$ by $\frac{a}{a+c}-\frac{b}{b+c}$. Ans. $\frac{(a^2+b^2)c}{(a+b)(a+c)(b+c)}$.

8. Multiply $3x^{\frac{7}{4}}-\frac{5xy}{x^{\frac{1}{4}}}$ by $x^{\frac{1}{4}}-\frac{7y}{x^{\frac{3}{4}}}$. Ans. $3x^2-26xy+35y^2$.

9. Multiply $\frac{a^2}{x^2}-\frac{ab}{2xy}+\frac{b^2}{y^2}$ by $\frac{3a^2}{x^2}-\frac{2ab}{5xy}+\frac{b^2}{y^2}$.

Ans. $\frac{3a^4}{x^4}-\frac{19a^3b}{10x^3y}+\frac{21a^2b^2}{5x^2y^2}-\frac{9ab^3}{10xy^3}+\frac{b^4}{y^4}$.

10. Multiply $\frac{a^2}{b^3}+\frac{2c^3d^4}{b^5}-\frac{7c^2}{2a^4b^3}$ by $\frac{a^2}{b^3}-\frac{2c^3d^4}{b^5}+\frac{7c^2}{2a^4b^3}$.

Ans. $\frac{a^4}{b^6}-\frac{4c^6d^8}{b^{10}}+\frac{14c^5d^4}{a^4b^8}-\frac{49c^4}{4a^8b^6}$.

11. Divide $\frac{3x}{2x-2}$ by $\frac{2x}{x-1}$. Ans. $\frac{3}{4}$.

12. Divide $x+\frac{2x}{x-3}$ by $x-\frac{2x}{x-3}$. Ans. $\frac{x-1}{x-5}$.

13. Divide $\frac{4a(a^2-x^2)}{3b(c^2-x^2)}$ by $\frac{a^2-ax}{bc+bx}$. Ans. $\frac{4(a+x)}{3(c-x)}$.

14. Divide $x^4-\frac{1}{x^4}$ by $x-\frac{1}{x}$. Ans. $x^3+\frac{1}{x^3}+x+\frac{1}{x}$.

15. Divide $a^6+\frac{1}{a^6}+a^4+\frac{1}{a^4}+a^2+\frac{1}{a^2}+2$ by $a^3+\frac{1}{a^3}+a+\frac{1}{a}$.

Ans. $a^3+\frac{1}{a^3}$.

16. Divide $\frac{a^2}{bc} - \frac{2a}{d} + \frac{ac}{be} + \frac{bc}{d^2} - \frac{c^2}{de}$ by $\frac{a}{b} - \frac{c}{d}$. Ans. $\frac{a}{c} - \frac{b}{d} + \frac{c}{e}$.

17. Divide $\frac{a^2x^3}{bd} + \frac{abx^2}{c^2d} - \frac{acx^2}{d^2} - \frac{b^2x}{cd^2} + \frac{a^2x}{bc} - \frac{a}{d}$ by $\frac{ax}{c} - \frac{b}{d}$; and verify the result.

18. Divide $a' + x$ by $a + x$ to 4 terms of quotient.

Ans. $\frac{a'}{a} + \frac{a-a'}{a^2} . x - \frac{a-a'}{a^3} . x^2 + \frac{a-a'}{a^4} . x^3 - \dots$

19. Divide a^2 by $a^2 + 2ax + x^2$. Ans. $1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \dots$

20. Divide $a - bx$ by $a + cx$ to 4 terms.

Ans. $1 - (b+c)\frac{x}{a} + (b+c)\frac{cx^2}{a^2} - (b+c)\frac{c^2x^3}{a^3} + \dots$

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1. FIND the value of $\left(x + \frac{2x}{x-3}\right) \div \left(x - \frac{2x}{x-3}\right)$, when $x = 5\frac{1}{2}$. Ans. 9.

2. Find the value of $\frac{x}{2} - \left\{\frac{2x-3}{3} - \frac{3x-1}{4}\right\} \div \frac{x-1}{2}$, when $x = 4\frac{1}{3}$.

Ans. $2\frac{5}{6}$.

3. Find the value of $ax + by$, when $x = \frac{cq - br}{aq - bp}$, and $y = \frac{ar - cp}{aq - bp}$.

Ans. c.

4. Find the value of $\frac{x+2a}{x-2a} + \frac{x+2b}{x-2b}$, (1) when $x = \frac{4ab}{a+b}$, and (2) hen $x = \frac{12ab}{a + b + \surd\{(a+b)^2 + 12ab\}}$. (1) Ans. 2. (2). Ans. 1.

5. Find the value of $\frac{a^n}{2na^n - 2nx} + \frac{b^n}{2nb^n - 2nx}$, when $x = \frac{a^n + b^n}{2}$.

Ans. $\frac{1}{n}$.

6. If two fractions are together equal to 1, shew that their differ- ice is the same as the difference of their squares.

7. If the difference of two fractions is equal to $\frac{p}{q}$, shew that p nes their sum is equal to q times the difference of their squares.

8. Two fractions are together equal to $\frac{a}{b}$, and the one exceeds the other by $\frac{c}{d}$; what are the fractions? Ans. $\frac{1}{2}\left(\frac{a}{b}+\frac{c}{d}\right)$, and $\frac{1}{2}\left(\frac{a}{b}-\frac{c}{d}\right)$.

9. Divide $\frac{83}{89}$ into two parts differing by $\frac{14}{19}$. Ans. $\frac{2823}{3382}$, $\frac{331}{3382}$.

INVOLUTION AND EVOLUTION.*

1. FIND the square and the cube of $3a^2bc^3$. Ans. $9a^4b^2c^6$, $27a^6b^3c^9$.

2. Find the square and the cube of $\frac{2}{5}a^3x^{m+2}y^{p-1}$.

(1) Ans. $\frac{4}{25}a^6x^{2m+4}y^{2p-2}$, (2) Ans. $\frac{8}{125}a^9x^{3m+6}y^{3p-3}$.

3. Find the cube of $\frac{1}{2}a-\frac{2}{3}b$. Ans. $\frac{1}{8}a^3-\frac{8}{27}b^3-\frac{1}{2}a^2b+\frac{2}{3}ab^2$.

4. Find the cube of $1+2x+3x^2+4x^3$.

Ans. $1+6x+21x^2+56x^3+111x^4+174x^5+219x^6+204x^7+144x^8+64x^9$.

5. Find the cube of $\sqrt{a}-\sqrt{b}$. Ans. $(a+3b)\sqrt{a}-(b+3a)\sqrt{b}$.

6. Find the cube of $\sqrt[3]{x}-\sqrt[3]{y}$. Ans. $x-y-3\sqrt[3]{xy}\,.\,(\sqrt[3]{x}-\sqrt[3]{y})$.

7 Find the cube of $x-\frac{1}{x}$. Ans. $x^3-\frac{1}{x^3}-3\left(x-\frac{1}{x}\right)$.

8. Find the cube of $x-\frac{1}{x}-1$. Ans. $x^3-\frac{1}{x^3}-3x^2-3\frac{1}{x^2}+5$.

9. Find the square of $a^{\frac{2}{3}}-a^{\frac{1}{3}}+1$. Ans $a^{\frac{4}{3}}-2a+3a^{\frac{2}{3}}-2a^{\frac{1}{3}}+1$.

10. Find the cube of $\frac{x}{a^2}-\frac{a^2}{x}$. Ans. $\frac{x^3}{a^6}-\frac{3x}{a^2}+\frac{3a^2}{x}-\frac{a^6}{x^3}$.

11. Find the cube of e^x-e^{-x}. Ans. $e^{3x}-e^{-3x}-3(e^x-e^{-x})$.

12. Find the cube of $x^{pq}-1$. Ans. $x^{3pq}-3x^{2pq}+3x^{pq}-1$.

13. If $x+\frac{1}{x}=p$, prove that $x^3+\frac{1}{x^3}=p^3-3p$.

14. Find the square of $1-\frac{1}{2}x+\frac{2}{3}x^2-\frac{3}{4}x^3+\frac{4}{5}x^4-$ &c.

Ans. $1-x+\frac{19}{12}x^2-\frac{13}{6}x^3+\dots$

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* Each Example in Involution with its Answer supplies also an Example in Evolution, and *vice versâ*.

15. Prove that $\left(x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \ldots\right)^2 + \left(1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \ldots\right)^2$ is equal to 1.

16. Find the square root of $9a^4b^2c^{-12}$, and of $16x^{2m+2}a^{-2n}$.
Ans. $3a^2bc^{-6}$, $4x^{m+1}a^{-n}$.

17. Find the square root of $4x^4 - 12x^3 + 25x^2 - 24x + 16$.
Ans. $2x^2 - 3x + 4$.

18. Find the square root of $9a^4 - 12a^3b + 34a^2b^2 - 20ab^3 + 25b^4$.
Ans. $3a^2 - 2ab + 5b^2$.

19. Find the square root of $x^6 + 4x^5 + 10x^4 + 20x^3 + 25x^2 + 24x + 16$.
Ans. $x^3 + 2x^2 + 3x + 4$.

20. Find the square root of $x^4 - 2x^3 + \frac{3x^2}{2} - \frac{x}{2} + \frac{1}{16}$. Ans. $x^2 - x + \frac{1}{4}$.

21. Find the square root of $\frac{25a^2b^2}{4} + \frac{c^4}{9} - \frac{5abc^2}{3}$. Ans. $\frac{5ab}{2} - \frac{c^2}{3}$.

22. Find the square root of $x^2 + \frac{a^2}{9} + \frac{b^2}{4} + \frac{2}{3}ax - \frac{1}{3}ab - bx$.
Ans. $x + \frac{1}{3}a - \frac{1}{2}b$.

23. Find the square root of $\frac{1051x^2}{25} - \frac{6x}{5} - \frac{14x^3}{5} + 9 + 49x^4$.
Ans. $7x^2 - \frac{x}{5} + 3$.

24. Find the square root of $a^2 - \frac{3a\sqrt{a}}{2} - \frac{3\sqrt{a}}{2} + \frac{41a}{16} + 1$.
Ans. $a - \frac{3\sqrt{a}}{4} + 1$.

25. Find the square root of $\frac{a^2}{b^2} + \frac{b^2}{a^2} - 2$. Ans. $\frac{a}{b} - \frac{b}{a}$.

26. Find the square root of $\frac{a^2}{b^2} + \frac{b^2}{a^2} + \frac{2a}{b} + \frac{2b}{a} + 3$. Ans. $\frac{a}{b} + \frac{b}{a} + 1$.

27. Find the square root of $a^{2m} - 4a^{m+n} + 4a^{2n}$. Ans. $a^m - 2a^n$.

28. Find the square root of $\frac{x^2}{y^2}\left(\frac{x^2}{4y^2} + 1\right) + \frac{4y^2}{x^2}\left(\frac{y^2}{x^2} + 1\right) + 3$.
Ans. $\frac{x^2}{2y^2} + \frac{2y^2}{x^2} + 1$.

29. Find the square root of $\frac{x^2}{y^2} + \frac{y^2}{x^2} - \left(\frac{x}{y} + \frac{y}{x}\right)\sqrt{2} + 2\frac{1}{2}$.
Ans. $\frac{x}{y} + \frac{y}{x} - \frac{1}{\sqrt{2}}$.

30. Find the cube root of $a^6-6a^5+15a^4-20a^3+15a^2-6a+1$.

Ans. a^2-2a+1.

31. Find the cube root of $\frac{a^6}{b^2}-6a^4+12a^2b^2-8b^4$. Ans. $\frac{a^2}{b^{\frac{2}{3}}}-2b^{\frac{4}{3}}$.

32. Find the cube root of $\frac{a^3c^3}{b^3}x^6-\frac{3a^2c}{b}x^5+\frac{3ab}{c}x^4-\frac{b^3}{c^3}x^3$.

Ans. $\frac{ac}{b}x^2-\frac{b}{c}x$.

33. Find the square root of $(x+x^{-1})^2-4(x-x^{-1})$. Ans. $x-x^{-1}-2$.

34. Find the square root of $9a^{2m}+6a^{3m+1}+25c^{2m-4}-30a^mc^{m-2}+a^{4m+2}$ $-10a^{3m+1}c^{m-2}$. Ans. $3a^m-5c^{m-2}+a^{2m+1}$.

35. Find the cube root of $(a+1)^{6m}x^3+6ca^p(a+1)^{4m}x^2+12c^2a^{2p}(a+1)^{2m}x$ $+8c^3a^{3p}$. Ans. $(a+1)^{2m}x+2ca^p$.

36. Find the 6th root of $a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20$.

Ans. $a-\frac{1}{a}$.

37. Find the 4th root of $a^8-\frac{3m^2}{n}a^7+\frac{27m^4}{8n^2}a^6-\frac{27m^6}{16n^3}a^5+\frac{81m^8}{25n^4}a^4$.

Ans. $a^2-\frac{3m^2}{4n}a$.

38. Find the square root (without the aid of the common rule) of $4\{a^2-b^2)cd+(c^2-d^2)ab\}^2+\{(a^2-b^2)(c^2-d^2)-4abcd\}^2$. Ans. $(a^2+b^2)(c^2+d^2)$.

39. Find the coefficient of x^8 in the square, and in the cube, of $1.-2x+3x^2+4x^4-x^7$. (1) Ans. 20. (2) Ans. 168.

40. Find the coefficient of x^2 in the square of $(x+a)(x+b)(x+c)$.

Ans. $a^2b^2+a^2c^2+b^2c^2+4abc(a+b+c)$.

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1. If two numbers differ by a unit, prove that the difference of their squares is the sum of the two numbers.

2. Shew that the sum of the cubes of any three consecutive integers is divisible by three times the second of them.

3. In the extraction of the square root according to the common rule, shew that, if the remainder is not greater than twice the root obtained, the last digit in the root is correctly found.

4. Prove that $x^4+px^3+qx^2+rx+s$ is a perfect square, if $p^2s=r^2$, and $q=\frac{p^2}{4}+2\sqrt{s}$.

5. Find the relations subsisting between a, b, c, d, when ax^3+bx^2+cx+d is a complete cube. Ans. $ac^3=db^3$, and $b^2=3ac$.

SURDS.

1. **Reduce** to simplest form $\sqrt{150}$, $\sqrt{1805}$, $\sqrt[3]{320}$, $\sqrt[3]{2808}$, $\sqrt[4]{144}$.

Ans. $5\sqrt{6}$, $19\sqrt{5}$, $4\sqrt[3]{5}$, $6\sqrt[3]{13}$, $2\sqrt[4]{9}$.

2. Reduce to simplest form $\sqrt[4]{768}$, $\sqrt[5]{608}$, $\sqrt[7]{640}$, $\sqrt[n]{a^{mn}b^n}$.

Ans. $4\sqrt[4]{3}$, $2\sqrt[5]{19}$, $2\sqrt[7]{5}$, $ab\sqrt[n]{a^m}$.

3. Reduce to simplest form $\sqrt{a^4b^3c}$, $\sqrt[3]{a^5b^3}$, $\sqrt{a^{4n}b^{2p}c^3}$, $\sqrt{a^3-a^2b}$.

Ans. $a^2b\sqrt{bc}$, $ab\sqrt[3]{a^2}$, $a^{2n}b^pc\sqrt{c}$, $a\sqrt{a-b}$.

4. Reduce to simplest form $\sqrt{ax^2-6ax+9a}$, $\sqrt{(x^2-y^2)(x+y)}$.

Ans. $(x-3)\sqrt{a}$, $(x+y)\sqrt{x-y}$.

Simplify the following Surds :—

5. $\sqrt{48ab^2}+b\sqrt{75a}+\sqrt{3a(a-9b)^2}$. Ans. $a\sqrt{3a}$.

6. $6\sqrt[6]{4a^2}+2\sqrt[3]{2a}+\sqrt[9]{8a^3}$. Ans. $9\sqrt[3]{2a}$.

7. $2\sqrt{8}-7\sqrt{18}+5\sqrt{72}-\sqrt{50}$. Ans. $8\sqrt{2}$.

8. $2\sqrt{\frac{5}{3}}+\sqrt{60}-\sqrt{15}+\sqrt{\frac{3}{5}}$. Ans. $\frac{28}{15}\sqrt{15}$.

9. $\frac{8\sqrt{3}}{4}-\frac{1}{2}\sqrt{12}+4\sqrt{27}-2\sqrt{\frac{3}{16}}$. Ans. $\frac{25}{2}\sqrt{3}$.

10. $7\sqrt[3]{54}+3\sqrt[3]{16}+\sqrt[3]{2}-5\sqrt[3]{128}$. Ans. $8\sqrt[3]{2}$.

11. $\sqrt[3]{81}-\sqrt[3]{-512}+\sqrt[3]{192}$. Ans. $8+7\sqrt[3]{3}$.

12. $(\sqrt[3]{54}+\sqrt[3]{250}+\sqrt[3]{128})\times(\sqrt[3]{54}+\sqrt[3]{250}-\sqrt[3]{128})$. Ans. $48\sqrt[3]{4}$.

13. $\sqrt{18a^5b^3}+\sqrt{50a^3b^3}$. Ans. $(3a^2b+5ab)\sqrt{2ab}$.

14. $\sqrt{25a^2b}+\sqrt{144a^2b}-\sqrt{289a^2b}$. Ans. 0.

15. $3b\sqrt[3]{2a^5b^2}-7\sqrt[3]{2a^5b^5}+8a\sqrt[3]{2a^2b^5}$. Ans. $4ab\sqrt[3]{2a^2b^2}$.

16. $\frac{ab}{b-c}\pm\sqrt{\frac{a^2b^2}{(b-c)^2}-\frac{a^2b}{b-c}}$. Ans. $\frac{a\sqrt{b}}{\sqrt{b}\mp\sqrt{c}}$.

17. $\sqrt{\frac{ab^3}{c^2}}+\frac{1}{2c}\sqrt{a^3b-4a^2b^2+4ab^3}$. Ans. $\frac{a}{2c}\sqrt{ab}$.

18. $5\sqrt{3}\times 7\sqrt{\frac{8}{3}}\times\sqrt{2}$. Ans. 140.

19. $\sqrt[3]{4} \times 7\sqrt[3]{6} \times \frac{1}{2}\sqrt[3]{5}$. Ans. $7\sqrt[3]{15}$.

20. $(3+\sqrt{5}) \times (2-\sqrt{5})$. Ans. $1-\sqrt{5}$.

21. $\left(-5-\sqrt{\frac{3}{4}}\right) \times \left(-5+\sqrt{\frac{3}{4}}\right)$. Ans. $24\frac{1}{4}$.

22. $\left(4\sqrt{\frac{7}{3}}+5\sqrt{\frac{1}{2}}\right) \times \left(\sqrt{\frac{7}{3}}+2\sqrt{\frac{1}{2}}\right)$. Ans. $14\frac{1}{3}+13\sqrt{\frac{7}{6}}$.

23. $5\sqrt{2} \times 3\sqrt{4+6\sqrt{2}}$. Ans. $30\sqrt{2+3\sqrt{2}}$.

24. $(\sqrt[3]{4}+\sqrt[3]{9}+\sqrt[3]{48}) \times (\sqrt[3]{2}+\sqrt[3]{3})$. Ans. $5+3\sqrt[3]{18}+3\sqrt[3]{12}$.

25. $(\sqrt[3]{12+\sqrt{19}})\,(\sqrt[3]{12-\sqrt{19}})$. Ans. 5.

26. $\sqrt[3]{40}-\frac{1}{2}\sqrt[3]{320}+\sqrt[3]{135}$. Ans. $3\sqrt[3]{5}$.

27. $\frac{\sqrt{3}+1}{2-\sqrt{3}}$, $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}$. Ans. $5+3\sqrt{3}$, $4+\sqrt{2}$.

28. $\frac{\frac{1}{2}\cdot\sqrt{\frac{1}{2}}}{\sqrt{2}+3\sqrt{\frac{1}{2}}}$, $(\sqrt{2}+\sqrt{3})^3$. Ans. $\frac{1}{10}$, $11\sqrt{2}+9\sqrt{3}$.

29. $\frac{\frac{1}{4}(\sqrt{5}\mp 1)}{\frac{1}{2\sqrt{2}}\sqrt{5\pm\sqrt{5}}}$. Ans. $\sqrt{1\mp\frac{2}{5}\sqrt{5}}$.

30. $\frac{\frac{1}{2\sqrt{2}}\cdot\sqrt{5\mp\sqrt{5}}}{\frac{1}{4}(\sqrt{5}\pm 1)}$. Ans. $\sqrt{5\mp 2\sqrt{5}}$.

31. $\sqrt[3]{\sqrt[4]{8a^3}}$, $\sqrt[2]{\sqrt[5]{9a^2}}$, $(\sqrt[5]{a^4b})^2$. Ans. $\sqrt[4]{2a}$, $\sqrt[5]{3a}$, $a\sqrt[5]{a^3b^2}$.

32. $\sqrt[5]{\sqrt[3]{a^2}}\cdot\sqrt[6]{\sqrt[4]{a^9}}$, $\frac{b}{\sqrt{a}}\sqrt[3]{ac}\cdot\frac{\sqrt[4]{c^3}}{\sqrt{b}}$. Ans. $\sqrt{a}\cdot\sqrt[120]{a}$, $c\sqrt[12]{\frac{b^6c}{a^2}}$.

33. $\sqrt[m]{a^{2m-n}b^{5m+1}c^{3p}} \times \sqrt[m]{a^n b^{m-1} c^{m-3p}}$. Ans. a^2b^6c.

34. $\sqrt[4]{\frac{a}{b}} \times \sqrt{\frac{b}{a}}$, $\sqrt[6]{\frac{1}{a^3b^2}} \times \sqrt[5]{a^2}.\sqrt[3]{b}$. Ans. $\sqrt[4]{\frac{b}{a}}$, $\frac{1}{\sqrt[10]{a}}$.

35. $\dfrac{\sqrt{1-x}+\dfrac{1}{\sqrt{1+x}}}{1+\dfrac{1}{\sqrt{1-x^2}}}$. Ans. $\sqrt{1-x}$.

36. $a\sqrt{\dfrac{a^3b}{3a^2+6ac+3c^2}}-\dfrac{bc\sqrt{ab}}{a+c}$. Ans. $\left(\dfrac{a^2\sqrt{3}}{3}-bc\right)\dfrac{\sqrt{ab}}{a+c}$.

37. $\dfrac{2x^2}{(1-x^2)^{\frac{3}{2}}}-\dfrac{1}{(1-x^2)^{\frac{1}{2}}}$. Ans. $\dfrac{3x^2-1}{(1-x^2)^{\frac{3}{2}}}$.

38. $\dfrac{1}{\sqrt{2}}\cdot\dfrac{2b^2-2c^2-bc\sqrt{2}}{2b^2-c^2}$. Ans. $\dfrac{c\sqrt{2}-b}{c-b\sqrt{2}}$.

39. $\dfrac{1}{x+\sqrt{x^2-1}}+\dfrac{1}{x-\sqrt{x^2-1}}$. Ans. $2x$.

40. $\dfrac{\sqrt{18a^5b^3}+\sqrt{50a^3b^5}}{\sqrt{8ab^3}+\sqrt{2a^3b-8a^2b^2+8ab^3}}$. Ans. $3ab+5b^2$.

41. $\dfrac{\sqrt[3]{x^4}+\sqrt[3]{x^2y^2}-\sqrt[3]{x^3y}}{\sqrt[3]{x^4}+\sqrt[3]{xy^3}-\sqrt[3]{x^3y}-\sqrt[3]{y^4}}$. Ans. $\dfrac{\sqrt[3]{x^2}}{\sqrt[3]{x^2}-\sqrt[3]{y^2}}$.

42. $\dfrac{x}{x-7}\left(\sqrt[18]{\{3a^2x^3-63a^2x^2+441a^2x-1029a^2\}}\right)^6$. Ans. $x\sqrt[3]{3a^2}$.

43. $\sqrt{a^2+\sqrt[3]{a^4b^2}}+\sqrt{b^2+\sqrt[3]{a^2b^4}}$. Ans. $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$.

44. $\sqrt[4]{\left(\dfrac{a\sqrt{b}}{\sqrt[3]{ab}}\right)^3}$, $\sqrt{\left[\left\{\dfrac{a^{-2m}}{b^{-2n}}\right\}^{\frac{p}{m}}\right]^{\frac{q}{n}}}$. Ans. $\sqrt[8]{a^4b}$, $\left\{\dfrac{b^{\frac{1}{m}}}{a^{\frac{1}{n}}}\right\}^{pq}$.

45. $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$. Ans. $\dfrac{a}{x}+\sqrt{\dfrac{a^2}{x^2}-1}$.

46. $\sqrt{\dfrac{ay}{x}}\cdot\sqrt[3]{\dfrac{bx}{y^2}}\cdot\sqrt[6]{\dfrac{y^2}{b^2a^3}}$. Ans. $\sqrt[6]{\dfrac{y}{x}}$.

47. Multiply $x+\dfrac{p}{2}+\sqrt{q+\dfrac{p^2}{4}}$ by $x+\dfrac{p}{2}-\sqrt{q+\dfrac{p^2}{4}}$.
Ans. x^2+px-q.

48. Multiply $\dfrac{x}{b}\sqrt{\dfrac{a}{b}}+\sqrt{\dfrac{c}{d}}$ by $\dfrac{x}{b}\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{c}{d}}$. Ans. $\dfrac{ax^2}{b^3}-\dfrac{c}{d}$.

49. Multiply $\sqrt[3]{a^{-\frac{1}{2}}}+\sqrt{\sqrt[3]{a^{\frac{1}{2}}b}}$ by $\sqrt[3]{a^{-\frac{1}{2}}}-\sqrt[3]{\sqrt{a^{\frac{1}{2}}b}}$.
Ans. $\sqrt[3]{a^{-1}}-\sqrt[3]{a^{\frac{1}{2}}b}$.

50. Divide $a^3 - 2\sqrt[4]{a^2b^3} - a^2\sqrt[6]{a^3b^2} + 2b\sqrt[12]{b}$ by $\sqrt{a} - \sqrt[3]{b}$.
Ans. $a^2\sqrt{a} - 2\sqrt[4]{b^3}$.

51. Divide $\sqrt[12]{a^{10}b^9} - c\sqrt[10]{a^7}\cdot\sqrt[6]{b^5} - \frac{3}{2}a\sqrt[4]{b^3} + \frac{3abc}{2}\sqrt[20]{\frac{1}{a^4b^5}}$ by $\sqrt{ab} - \frac{3}{2}\sqrt[6]{a^4b^3}$.
Ans. $\sqrt[12]{a^4b^3} - c\sqrt[15]{a^3b^5}$.

52. Divide $x - a$ by $x^{\frac{1}{n}} - a^{\frac{1}{n}}$.
Ans. $x^{1-\frac{1}{n}} + a^{\frac{1}{n}}x^{1-\frac{2}{n}} + a^{\frac{2}{n}}x^{1-\frac{3}{n}} + \ldots\ldots + a^{1-\frac{2}{n}}x^{\frac{1}{n}} + a^{1-\frac{1}{n}}$.

53. Square $\frac{c}{2} + \frac{1}{2}\sqrt{a^2 - c^2}$.
Ans. $\frac{a^2}{4} + \frac{c}{2}\sqrt{a^2 - c^2}$.

54. Square the expression $\sqrt{\frac{a + \sqrt{a^2 - b}}{2}} + \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$.
Ans. $a + \sqrt{b}$.

55. Find the 5th root of $\frac{3^{10}a^{-5}(a+b)^5\cdot(2+x)^{-10}}{32c^5d^{-15}}$.
Ans. $\frac{9d^3(a+b)}{2ac(2+x)^2}$.

56. Find the 9th root of $2^{36}a^{45}b^9\cdot\frac{(a+b)^{27}}{a^9}$.
Ans. $16a^4b(a+b)^3$.

57. Find the square root of $-2 + a^{2\sqrt{2}} + a^{-2\sqrt{2}}$.
Ans. $a^{\sqrt{2}} - a^{-\sqrt{2}}$.

58. Find the square root of $\frac{a^2c}{b} - cf + 2ac\sqrt{-\frac{f}{b}}$.
Ans. $a\sqrt{\frac{c}{b}} + \sqrt{-cf}$.

59. Find the square root of $\sqrt[x]{a^{-\frac{2}{y}}} + \sqrt[r]{a^{\frac{1}{y}}b} + 2\sqrt[2]{\sqrt[r]{b}}\cdot\sqrt[2]{\sqrt[x]{\sqrt[y]{\sqrt[r]{a^{x-2r}}}}}$.
Ans. $\sqrt[x]{a^{-\frac{1}{y}}} + \sqrt[2]{\sqrt[r]{a^{\frac{1}{y}}b}}$.

60. Find the square root of
$x^m + \frac{1}{4}\sqrt[\frac{np}{2}]{b^px^{2n}} - \sqrt[n]{b}\cdot\sqrt[2p]{x^{mp+4}}$.
Ans. $x^{\frac{m}{2}} - \frac{1}{2}b^{\frac{1}{n}}x^{\frac{2}{p}}$.

Extract the square root of each of the following quantities:

61. $18 + 2\sqrt{77}$. Ans. $\sqrt{11} + \sqrt{7}$.

62. $94 - 42\sqrt{5}$. Ans. $7 - 3\sqrt{5}$.

63. $28 + 10\sqrt{3}$. Ans. $5 + \sqrt{3}$.

64. $13 + 2\sqrt{30}$. Ans. $\sqrt{10} + \sqrt{3}$.

65. $a + x + \sqrt{2ax + x^2}$. Ans. $\sqrt{a + \frac{x}{2}} + \sqrt{\frac{x}{2}}$.

66. Find the square root of $\frac{a^2}{4}+\frac{c}{2}\sqrt{a^2-c^2}$. Ans. $\frac{c}{2}+\frac{1}{2}\sqrt{a^2-c^2}$.

67. .... $x+y+z+2\sqrt{xz+yz}$. Ans. $\sqrt{x+y}+\sqrt{z}$.

68. $m\times(\sqrt[3]{7}-\sqrt[3]{5})$ is a rational quantity; find m.

Ans. $\sqrt[3]{49}+\sqrt[3]{35}+\sqrt[3]{25}$.

69. $m\times(3^{\frac{2}{3}}+5^{\frac{1}{2}})$ is a rational quantity; find m.

Ans. $5^{\frac{5}{2}}-25\,.\,3^{\frac{2}{3}}+5^{\frac{3}{2}}\,.\,3^{\frac{4}{3}}-45+5^{\frac{1}{2}}\,.\,3^{\frac{8}{3}}-3^{\frac{10}{3}}$.

70. $m\times(\sqrt{2}+\sqrt[3]{3})$ is rational; find m.

Ans. $2^{\frac{5}{2}}-4\,.\,3^{\frac{1}{3}}+2^{\frac{3}{2}}\,.\,3^{\frac{2}{3}}-6+2^{\frac{1}{2}}\,.\,3^{\frac{4}{3}}-3^{\frac{5}{3}}$.

71. $m\times(\sqrt[3]{9}+\sqrt{5})$ is rational; find m.

Ans. $3^{\frac{10}{3}}-3^{\frac{8}{3}}\,.\,5^{\frac{1}{2}}+45-5^{\frac{3}{2}}\,.\,3^{\frac{4}{3}}+25\,.\,3^{\frac{2}{3}}-5^{\frac{5}{2}}$.

72. Reduce $\dfrac{\sqrt[3]{5\cdot12}+\sqrt[3]{0\cdot03375}}{\sqrt[3]{80}-\sqrt[3]{0\cdot01}}$ to its equivalent simple decimal.

Ans. 0·5.

73. Reduce the following fractions to equivalent fractions with rational denominators:—

(1) $\dfrac{1+\sqrt{3}}{2\sqrt{2}-3\sqrt{3}}$, (2) $\dfrac{1+\sqrt{2}}{1+\sqrt{2}+\sqrt{3}}$, (3) $\dfrac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}}$,

(4) $\dfrac{\sqrt{10}}{\sqrt{2}-\sqrt[3]{3}}$, (5) $\dfrac{a-b}{\sqrt[4]{a}-\sqrt[4]{b}}$, (6) $\dfrac{n}{a+\sqrt[5]{b}}$.

(1) Ans. $-\frac{1}{19}\{3\sqrt{3}+2\sqrt{2}+2\sqrt{6}+9\}$,

(2) Ans. $\frac{1}{4}\{4+3\sqrt{2}-2\sqrt{3}-\sqrt{6}\}$, (3) Ans. $\sqrt[3]{40}+\sqrt[3]{20}+\sqrt[3]{10}$,

(4) Ans. $-\sqrt{10}\,.\,\{4\sqrt{2}+4\sqrt[3]{3}+2\sqrt{2}\,.\,\sqrt[3]{9}+6+3\sqrt{2}\,.\,\sqrt[3]{3}+3\sqrt[3]{9}\}$,

(5) Ans. $a^{\frac{3}{4}}+a^{\frac{1}{2}}b^{\frac{1}{4}}+a^{\frac{1}{4}}b^{\frac{1}{2}}+b^{\frac{3}{4}}$,

(6) Ans. $\dfrac{n\,(a^4-a^3b^{\frac{1}{5}}+a^2b^{\frac{2}{5}}-a\,b^{\frac{3}{5}}+b^{\frac{4}{5}}}{a^5+b}$.

74. Prove that $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$ is equal to $\sqrt{2}$.

75. Simplify $\dfrac{1}{\sqrt[3]{x^2}+\sqrt[3]{ax}+\sqrt[3]{a^2}}$. Ans. $\dfrac{\sqrt[3]{x}-\sqrt[3]{a}}{x-a}$.

76. Simplify $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$. Ans. $\frac{1}{12}\sqrt{30}+\frac{1}{4}\sqrt{2}+\frac{1}{6}\sqrt{3}$.

77. Find the value of $\dfrac{7\sqrt{5}}{\sqrt{11}+\sqrt{3}}$. Ans. 3·10.

78. Find the value of $\dfrac{3+2\sqrt{7}}{5\sqrt{12}-6\sqrt{6}}$. Ans. 3·159.

79. Simplify $\dfrac{\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}}}{\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}}}$. Ans. $\sqrt{5}+1-\sqrt{5+2\sqrt{5}}$.

80. Simplify $\sqrt{\dfrac{2-\sqrt{2+\sqrt{2}}}{2+\sqrt{2+\sqrt{2}}}}$. Ans. $\sqrt{2(2+\sqrt{2})}-\sqrt{2}-1$.

81. Simplify $\dfrac{2\sqrt{2+\sqrt{3}}}{4+\sqrt{6}-\sqrt{2}}$. Ans. $\sqrt{6}-\sqrt{2}+\sqrt{3}-2$.

82. Simplify the expressions

$$(b+\sqrt{2a-b^2})^4+(b-\sqrt{2a-b^2})^4;\quad (b+\sqrt{2a-b^2})^4-(b-\sqrt{2a-b^2})^4.$$

(1) Ans. $8(a^2+2ab^2-b^4)$; (2) Ans. $16ab\sqrt{2a-b^2}$.

83. Simplify $(a+b\sqrt{-1})^4+(a-b\sqrt{-1})^4$. Ans. $2a^4-12a^2b^2+2b^4$.

84. Prove that $5^{\frac{3}{4}}.\left\{\dfrac{\sqrt{5}\pm1}{2}\right\}^{\frac{1}{2}}$ is equivalent to $\dfrac{5}{2}\sqrt{2\pm\dfrac{2}{5}\sqrt{5}}$; and

$$5^{\frac{3}{4}}.\left\{\frac{\sqrt{5}\pm1}{2}\right\}^{\frac{5}{2}} \text{ to } \frac{5}{2}\sqrt{10\pm\frac{22}{5}\sqrt{5}}.$$

85. Find the value of $\dfrac{x^{\frac{1}{2}}+x}{1+x^{\frac{3}{2}}}$, when $x=\dfrac{a^2+a\sqrt{a^2+4}}{2}$.

Ans. $\dfrac{2a}{\sqrt{(a^2+1+a\sqrt{a^2+4})^2-1}-2a}$.

86. Simplify $\dfrac{n^3-3n+(n^2-1)\sqrt{n^2-4}-2}{n^3-3n+(n^2-1)\sqrt{n^2-4}+2}$. Ans. $\dfrac{n+1}{n-1}\sqrt{\dfrac{n-2}{n+2}}$.

87. Simplify $\left\{\dfrac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}}\right\}^{\frac{1}{2}}$. Ans. $\dfrac{x+\sqrt{4x^2+3}}{3\sqrt{x^2+1}}$.

88. Simplify $\dfrac{xy-\sqrt{\dfrac{1-x^2}{1+x^2}\cdot\dfrac{1-y^2}{1+y^2}}}{1+xy\sqrt{\dfrac{1-x^2}{1+x^2}\cdot\dfrac{1-y^2}{1+y^2}}}$. Ans. $\dfrac{2xy-\sqrt{(1-x^4)(1-y^4)}}{1+x^2+y^2-x^2y^2}$.

89. Simplify $\dfrac{1-ab+\sqrt{1+a^2}-a\sqrt{1+b^2}}{1-ab+\sqrt{1+b^2}-b\sqrt{1+a^2}}$. Ans. $\dfrac{1-a+\sqrt{1+a^2}}{1-b+\sqrt{1+b^2}}$.

90. Find the value of $\sqrt{\dfrac{n^2-1}{n^2+1}}$, when $n^2=\dfrac{1+\sqrt{3}+\sqrt{2\sqrt{3}}}{2}$.

Ans. $\sqrt{\dfrac{\sqrt{3}-1}{\sqrt{2\sqrt{3}}}}$.

91. Find the value of $\dfrac{1-ax}{1+ax}\cdot\sqrt{\dfrac{1+bx}{1-bx}}$, when $x=\dfrac{1}{a}\sqrt{\dfrac{2a}{b}-1}$.

Ans. 1.

92. If $x=\sqrt[3]{-\dfrac{r}{2}\pm\sqrt{\dfrac{r^2}{4}-\dfrac{q^3}{27}}}$, find the value of $x^6+rx^3+\dfrac{q^3}{27}$.

Ans. 0.

93. Find the value of $\dfrac{\sqrt{a+x}+\sqrt{a-x}}{\sqrt{a+x}-\sqrt{a-x}}$, when $x=\dfrac{2ab}{b^2+1}$. Ans. b.

94. Find the value of $2(uv-\sqrt{1-u^2}.\sqrt{1-v^2})$, when $2u=x+x^{-1}$, and $2v=y+y^{-1}$. Ans. $xy+(xy)^{-1}$.

95. Find the value of $\dfrac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$, when $x=\dfrac{1}{2}\left\{\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{b}{a}}\right\}$.

Ans. $a+b$.

96. If $x=\dfrac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}-1)$, and $y=\dfrac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}+1)$, find the value of x^5+y^5.

Ans. 2.

97. If $x=\left(\dfrac{a+b}{a-b}\right)^{\frac{2pq}{q-p}}$, find the value of $\dfrac{1}{2}\cdot\dfrac{a^2-b^2}{a^2+b^2}(\sqrt[p]{x}+\sqrt[q]{x})$.

Ans. $\left(\dfrac{a+b}{a-b}\right)^{\frac{q+p}{q-p}}$.

EQUATIONS.

1. If $\dfrac{9}{2x}-4=\dfrac{2}{3}$, $x=\dfrac{27}{28}$.

2. If $13\tfrac{3}{4}-\dfrac{x}{2}=2x-8\tfrac{3}{4}$, $x=9$.

3. If $\dfrac{3x}{4}+\dfrac{7x}{15}+\dfrac{11x}{6}=366$, $x=120$.

4. If $\dfrac{1}{x}+\dfrac{1}{2x}-\dfrac{1}{3x}=\dfrac{7}{3}$, $x=\dfrac{1}{2}$.

5. If $5x-\frac{2x-1}{3}+1=3x+\frac{x+2}{2}+7,$ $x=8.$

6. If $\frac{x+6}{4}-\frac{16-3x}{12}=\frac{25}{6},$ $x=8.$

7. If $20x-50(3x-4)=200-20(4x-5)-(3x-4)50,$ $x=3.$

8. If $4(5x-3)-64(3-x)-3(12x-4)=96,$ $x=6.$

9. If $10\left(x+\frac{1}{2}\right)-6x\left(\frac{1}{x}-\frac{1}{3}\right)=23,$ $x=2.$

10. If $\frac{1}{12}x-\frac{1}{8}(8-x)-\frac{1}{4}(5+x)+\frac{11}{4}=0,$ $x=12.$

11. If $\frac{1}{7}\left(x-\frac{1}{2}\right)-\frac{1}{5}\left(\frac{2}{3}-x\right)=1\frac{13}{30},$ $x=4\frac{7}{9}.$

12. If $3x-4-\frac{4}{5}\cdot\frac{7x-9}{3}=\frac{4}{5}\left(6+\frac{x-1}{3}\right),$ $x=7\frac{1}{13}.$

13. If $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1},$ $x=3.$

14. If $\frac{5x+3}{x-1}+\frac{2x-3}{2x-2}=9,$ $x=3\frac{1}{2}.$

15. If $\frac{10x+17}{18}-\frac{12x+2}{11x-8}=\frac{5x-4}{9},$ $x=4.$

16. If $\frac{1}{x-1}-\frac{2}{x+7}=\frac{1}{7(x-1)},$ $x=7.$

17. If $\frac{4x}{5-x}-\frac{20-4x}{x}=\frac{15}{x},$ $x=3\frac{2}{11}.$

18. If $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5},$ $x=20.$

19. If $\frac{1}{3}-\frac{7x-1}{6\frac{1}{2}-3x}=\frac{8}{3}\cdot\frac{x-\frac{1}{2}}{x-2},$ $x=1\frac{1}{13}.$

20. If $\frac{7x+5}{23}+\frac{9x-1}{10}-\frac{x-9}{5}+\frac{2x-3}{15}=23\frac{1}{3},$ $x=19.$

21. If $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2},$ $x=72.$

22. If $\frac{x}{2} - \frac{\frac{1}{3}(2x-3) - \frac{1}{4}(3x-1)}{\frac{1}{2}(x-1)} = \frac{3}{2} \cdot \frac{x^2+2}{3x-2}$, $x = 4\frac{1}{3}$.

23. If $3\frac{1}{3} \cdot \left\{ 28 - \left(\frac{x}{8} + 24 \right) \right\} = 3\frac{1}{2} \cdot \left\{ 2\frac{1}{3} + \frac{x}{4} \right\}$, $x = 4$.

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24. If $\frac{x}{a} + \frac{x}{b} = c$, $x = \frac{abc}{a+b}$.

25. If $\frac{ax}{b} + \frac{cx}{f} + g = qx + \frac{cx}{f} + h$, $x = b \cdot \frac{h-g}{a-bq}$.

26. If $\frac{a}{bx} + \frac{b}{ax} = a^2 + b^2$, $x = \frac{1}{ab}$.

27. If $a + x + \sqrt{2ax + x^2} = b$, $x = \frac{(a-b)^2}{2b}$.

28. If $a + x + \sqrt{a^2 + x^2} = b$, $x = \frac{b}{2}\left\{ 1 - \frac{a}{b-a} \right\}$.

29. If $\sqrt{x} + \sqrt{a - \sqrt{ax + x^2}} = \sqrt{a}$, $x = \frac{9a}{16}$.

30. If $a + x + \sqrt{a^2 + bx + x^2} = b$, $x = b \cdot \frac{b-2a}{3b-2a}$.

31. If $\sqrt{a-x} + 2\sqrt{a+x} = \sqrt{a - x + \sqrt{ax + x^2}}$, $x = \frac{64a}{1025}$.

32. If $\sqrt{4a + x} = 2\sqrt{b + x} - \sqrt{x}$, $x = \frac{(a-b)^2}{2a-b}$.

33. If $\sqrt{1 + x + x^2} = a - \sqrt{1 - x + x^2}$, $x = \frac{a}{2}\sqrt{1 - \frac{3}{a^2-1}}$.

34. If $\sqrt{a+x} + \sqrt{a-x} = b\sqrt[4]{a^2 - x^2}$, $x = \frac{ab}{b^2-2}\sqrt{b^2-4}$.

35. If $\frac{a}{b+x} + \frac{a}{b-x} = c$, $x = b\sqrt{1 - \frac{2a}{bc}}$.

36. If $\frac{1}{a}\sqrt{a+x} + \frac{1}{x}\sqrt{a+x} = \frac{1}{b}\sqrt{x}$, $x = \frac{ab^{\frac{2}{3}}}{a^{\frac{2}{3}} - b^{\frac{2}{3}}}$.

37. If $\sqrt{a^2 + x^2} + \sqrt{a^2 - x^2} = b$, $x = b\sqrt[4]{\left(\frac{a}{b}\right)^2 - \left(\frac{1}{2}\right)^2}$.

38. If $\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1$, $\qquad x=\frac{16}{25}$.

39. If $1+\sqrt{1+x}-\sqrt{1+x+\sqrt{1-x}}=0$, $\qquad x=-\frac{24}{25}$.

40. If $x+\sqrt{a^2+x^2}=\frac{na^2}{\sqrt{a^2+x^2}}$, $\qquad x=a\cdot\frac{n-1}{\sqrt{2n-1}}$.

41. If $\sqrt{4a+x}+\sqrt{a+x}=2\sqrt{x-2a}$, $\qquad x=\frac{17a}{8}$.

42. If $\sqrt{1+x}+\sqrt{1+x+\sqrt{1-x}}=\sqrt{1-x}$, $\qquad x=-\frac{24}{25}$.

43. If $\sqrt{x+\sqrt{x}}-\sqrt{x-\sqrt{x}}=a\sqrt{\frac{x}{x+\sqrt{x}}}$, $\qquad x=\frac{1}{4}\left\{1-a+\frac{1}{1-a}\right\}^2$.

44. If $ax+\sqrt{a^2x^2+b^2}=\sqrt{b^2+ax\sqrt{4b^2+x^2}}$, $\qquad x=\frac{8ab}{\sqrt{1-16a^2}}$.

45. If $\frac{a^2-x^2}{a+\sqrt{a^2+x^2}}=b-\sqrt{a^2+x^2}$, $\qquad x=\frac{a^2}{a-b}\sqrt{3-\frac{2b}{a}}$.

46. If $\frac{ax-b^2}{\sqrt{ax}+b}=\frac{\sqrt{ax}-b}{n}-c$, $\qquad x=\frac{1}{a}\left\{b-\frac{nc}{n-1}\right\}^2$.

47. If $\sqrt{1+x+x^2}+\sqrt{1-x+x^2}=mx$, $\qquad x=\frac{2}{m}\sqrt{\frac{m^2-1}{m^2-4}}$.

48. If $\sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2}$, $\qquad x=\sqrt{\frac{a^2-1}{a^2+3}}$.

49. If $\sqrt[m]{a+x}=\sqrt[2m]{x^2+8ax+b^2}$, $\qquad x=\frac{a^2-b^2}{6a}$.

50. If $\sqrt{x^2+2ax}+\sqrt{x^2-2ax}=\frac{nax}{\sqrt{x^2+2ax}}$, $\qquad x=a\left\{\frac{n-2}{2}+\frac{2}{n-2}\right\}$.

51. If $\frac{ax-1}{\sqrt{ax}+1}=4+\frac{\sqrt{ax}-1}{2}$, $\qquad x=\frac{81}{a}$.

52. If $\sqrt{(2a+x)^2+b^2}+\sqrt{(2a-x)^2+b^2}=2a$, $\qquad x=\sqrt{a^2+\frac{b^2}{3}}$.

53. If $\sqrt{a+\sqrt{a^2-x^2}}+\sqrt{a-\sqrt{a^2-x^2}}=n\sqrt{\frac{a+x}{a+\sqrt{a^2-x^2}}}$,

$\qquad x=n\sqrt{a-\frac{n^2}{4}}$.

54. If $\dfrac{1+x^3}{(1+x)^2}+\dfrac{1-x^3}{(1-x)^2}=a$, $x=\sqrt{\dfrac{a-2}{a+4}}$.

55. If $\sqrt{(1+x)^2-ax}+\sqrt{(1-x)^2+ax}=x$, $x=2\sqrt{\dfrac{(a-1)(a-3)}{3}}$.

56. If $\dfrac{1}{\sqrt{1-x}+1}+\dfrac{1}{\sqrt{1+x}-1}=\dfrac{1}{x}$, $x=\dfrac{1}{2}\sqrt{3}$.

57. If $a+b=\dfrac{2a\sqrt{1+x^2}}{x+\sqrt{1+x^2}}$, $x=\dfrac{1}{2}\left\{\sqrt{\dfrac{a}{b}}-\sqrt{\dfrac{b}{a}}\right\}$.

58. If $a(1-x)+\sqrt{1+a^2}=a\sqrt{1+x^2}$, $x=\sqrt{1+a^2}\cdot\dfrac{1-a^2+a\sqrt{1+a^2}}{2a}$.

59. If $\dfrac{1+\sqrt{x^2-1}}{1+2a\sqrt{x^2-1}}=\dfrac{\sqrt{x^2-1}-1}{x^2-2}$, $x=\sqrt{1+4(a-1)^2}$.

60. If $\sqrt{\dfrac{3a}{4}-x}+\sqrt{3ax-x}=\dfrac{3a}{2}\sqrt{1-4x}$, $x=\dfrac{1}{12}\cdot\dfrac{3a-1}{a+1}$.

61. If $\sqrt[3]{1+x}+\sqrt[3]{1-x}=\sqrt[3]{2}$, $x=1$.

62. If $\dfrac{1+x}{1+x+\sqrt{1+x^2}}+\dfrac{1-x}{1-x+\sqrt{1+x^2}}=a$, $x=\sqrt{(2-a)^2-1}$.

63. If $\sqrt{1+x+x^2}+\sqrt{1-x+x^2}=a$, $x=\dfrac{a}{2}\sqrt{\dfrac{a^2-4}{a^2-1}}$.

64. If $\sqrt{\{a^2x^2+b\sqrt{(abx+4a^2x^2+b\sqrt{2abx+5a^2x^2})}\}}=ax+b$,

$x=-\dfrac{b}{2a}$.

65. If $\dfrac{1+x+\sqrt{2x+x^2}}{1-x+\sqrt{2x+x^2}}=1-ax$, $x=\dfrac{(a+2)^2}{4a(a+1)}$.

66. If $\sqrt{1+a}\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{1-a}\left(\dfrac{1-x}{1+x}\right)^{\frac{1}{4}}=2\sqrt[4]{1-a^2}$, $x=-a$.

67. If $\sqrt{(1+a)^2+(1-a)x}+\sqrt{(1-a)^2+(1+a)x}=2a$, $x=8$.

68. If $\dfrac{1+x-\sqrt{2x+x^2}}{1+x+\sqrt{2x+x^2}}=a\cdot\dfrac{\sqrt{2+x}+\sqrt{x}}{\sqrt{2+x}-\sqrt{x}}$, $x=\dfrac{1}{2}\left\{\sqrt[3]{a}+\dfrac{1}{\sqrt[3]{a}}\right\}-1$.

69. If $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=b$, $x=a\cdot\dfrac{(1-\sqrt{b})^2}{1+b}$.

70. If $\sqrt{a+x}+\sqrt{a-x}=\sqrt[4]{a^2+x^2}+\sqrt[4]{a^2-x^2}$, $x=\dfrac{2a}{3}\cdot\sqrt[4]{5}$.

71. If $\dfrac{a+x}{\sqrt{a}+\sqrt{a+x}} + \dfrac{a-x}{\sqrt{a}+\sqrt{a-x}} = \sqrt{a}$, $\qquad x = \dfrac{a}{2} \times \sqrt{3}$, or 0.

72. If $\dfrac{\sqrt{a}-\sqrt{a-\sqrt{a^2-ax}}}{\sqrt{a}+\sqrt{a-\sqrt{a^2-ax}}} = b$, $\qquad x = a\left\{1-\left(\dfrac{2\sqrt{b}}{b+1}\right)^4\right\}$.

73. If $\dfrac{\sqrt{1+x}-1}{\sqrt{1-x}+1} + \dfrac{\sqrt{1-x}+1}{\sqrt{1+x}-1} = a$, $\qquad x = \dfrac{4}{a}\sqrt{1-\left(\dfrac{2}{a}\right)^2}$.

74. If $\dfrac{\sqrt{a^2+x^2}-a}{\sqrt{a^2-x^2}+a} = b$, $\qquad x = \dfrac{2a\sqrt{b}}{1+b^2}\sqrt{1-b^2}$.

75. If $\dfrac{1-ax}{1+ax}\sqrt{\dfrac{1+bx}{1-bx}} = 1$, $\qquad x = \dfrac{1}{a}\sqrt{\dfrac{2a}{b}-1}$.

76. If $(1+x)\sqrt{1+a} + (1-x)\sqrt{1-a} = 2\sqrt{1+x^2}$,

$x = \dfrac{1}{a} - \sqrt{\left(\dfrac{1}{a}\right)^2 - 1}$.

77. If $\sqrt[4]{x^4-1} + x\sqrt{x^4-1} = x^3$, $\qquad x = \sqrt[4]{\dfrac{1}{2}+\dfrac{1}{2}\sqrt{2}}$.

78. If $\sqrt{x} + 2\sqrt{2a-x} = \sqrt{x+\sqrt{2a^2-3ax+x^2}}$,

$x = 2a$, or a, or $\dfrac{961a}{1025}$.

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1. If $\left.\begin{array}{l} ax = by, \\ \text{and } x + y = c, \end{array}\right\}$ $\qquad x = \dfrac{bc}{a+b},\ y = \dfrac{ac}{a+b}$.

2. If $\left.\begin{array}{l} ax + by = m, \\ \text{and } a'x + b'y = m', \end{array}\right\}$ $\qquad x = \dfrac{b'm - bm'}{ab' - a'b},\ y = \dfrac{am' - a'm}{ab' - a'b}$.

3. If $\left.\begin{array}{l} \dfrac{x}{9} + \dfrac{y}{8} = 43, \\ \text{and } \dfrac{x}{8} + \dfrac{y}{9} = 42, \end{array}\right\}$ $\qquad \left\{\begin{array}{l} x = 144, \\ y = 216. \end{array}\right.$

4. If $\left.\begin{array}{l} \dfrac{m}{x} + \dfrac{n}{y} = a, \\ \text{and } \dfrac{n}{x} + \dfrac{m}{y} = b, \end{array}\right\}$ $\qquad \left\{\begin{array}{l} x = \dfrac{m^2-n^2}{ma-nb}, \\ y = \dfrac{m^2-n^2}{mb-na}. \end{array}\right.$

5. If $\sqrt{ax} + \sqrt{by} = \dfrac{1}{2}(x+y) = a + b$, $\qquad \left\{\begin{array}{l} x = (\sqrt{a} \pm \sqrt{b})^2, \\ y = (\sqrt{a} \mp \sqrt{b})^2. \end{array}\right.$

6. If $axy = c(bx + ay)$, and $bxy = c(ax - by)$, $\quad \left\{\begin{aligned} x &= \frac{a^2 + b^2}{a^2 - b^2}\cdot c, \\ y &= \frac{a^2 + b^2}{2ab}\cdot c. \end{aligned}\right.$

7. If $(x+5)(y+7) = (x+1)(y-9) + 112$, and $2x + 10 = 3y + 1$, $\quad \left\{\begin{aligned} x &= 3, \\ y &= 5. \end{aligned}\right.$

8. If $a(x^2+y^2) - b(x^2-y^2) = 2a$, and $(a^2-b^2)(x^2-y^2) = 4ab$, $\quad \left\{\begin{aligned} x &= \sqrt{\frac{a+b}{a-b}}, \\ y &= \sqrt{\frac{a-b}{a+b}}. \end{aligned}\right.$

9. If $\dfrac{2x+y}{9} + \dfrac{7y+6x+11}{18} = \dfrac{19}{2} - \dfrac{5x-17}{6}$, and $\dfrac{3}{7}\{5x + 3y + 2\} = \dfrac{1}{2}\{9y + 6\}$, $\quad \left\{\begin{aligned} x &= 7, \\ y &= 4. \end{aligned}\right.$

10. If $\dfrac{3x-5y}{3} - \dfrac{2x-8y-9}{12} = \dfrac{y}{2} + \dfrac{1}{3} + \dfrac{1}{4}$, and $3\tfrac{1}{2}\cdot\left\{\dfrac{x}{7} + \dfrac{y}{4} + 1\tfrac{1}{3}\right\} = 3\tfrac{1}{3}\cdot\left\{4x - \dfrac{y}{8} - 24\right\}$, $\quad \left\{\begin{aligned} x &= 7, \\ y &= 4. \end{aligned}\right.$

11. If $2{\cdot}4x + 0{\cdot}32y - \dfrac{0{\cdot}36x - 0{\cdot}05}{0{\cdot}5} = 0{\cdot}8x + \dfrac{2{\cdot}6 + 0{\cdot}005y}{0{\cdot}25}$, and $\dfrac{0{\cdot}04y + 0{\cdot}1}{0{\cdot}3} = \dfrac{0{\cdot}07x - 0{\cdot}1}{0{\cdot}6}$, $\quad \left\{\begin{aligned} x &= 10, \\ y &= 5. \end{aligned}\right.$

12. If $\sqrt{y} - \sqrt{a-x} = \sqrt{y-x}$, and $2\sqrt{y-x} + 2\sqrt{a-x} = 5\sqrt{a-x}$, $\quad \left\{\begin{aligned} x &= \frac{4}{5}a, \\ y &= \frac{5}{4}a. \end{aligned}\right.$

13. If $\dfrac{x}{a} + \dfrac{y}{b} = 1 - \dfrac{x}{c}$, and $\dfrac{y}{a} + \dfrac{x}{b} = 1 + \dfrac{y}{c}$, $\quad \left\{\begin{aligned} x &= \frac{(ab + ac - bc)\,abc}{a^2b^2 + a^2c^2 - b^2c^2}, \\ y &= \frac{(ac - ab - bc)\,abc}{a^2b^2 + a^2c^2 - b^2c^2}. \end{aligned}\right.$

14. If $(a^2 - b^2)(3x + 5y) = (4a - b)2ab$, and $a^2x - \dfrac{ab^2c}{a+b} + (a + b + c)by = b^2x + (a + 2b)ab$, $\quad \left\{\begin{aligned} x &= \frac{ab}{a-b}, \\ y &= \frac{ab}{a+b}. \end{aligned}\right.$

15. If $$\left.\begin{array}{l} y+\frac{1}{2}x=41, \\ x+\frac{1}{4}z=20\frac{1}{2}, \\ y+\frac{1}{5}z=34, \end{array}\right\} \qquad \left\{\begin{array}{l} x=18, \\ y=32, \\ z=10. \end{array}\right.$$

16. If $$\left.\begin{array}{l} x+\frac{1}{2}(y+z)=102, \\ y+\frac{1}{3}(z+x)=78, \\ z+\frac{1}{4}(x+y)=61, \end{array}\right\} \qquad \left\{\begin{array}{l} x=62, \\ y=46, \\ z=34. \end{array}\right.$$

17. If $\frac{x+y}{z}=\frac{44}{17}$, $\frac{x-y}{z}=\frac{20}{17}$, and $\frac{x^2-y^2}{z}=\frac{3520}{17}$,

$x=128,\quad y=48,\quad z=68.$

18. If $4x-5y+mz=7x-11y+nz=x+y+pz=3$,

$x=2,\quad y=1,\quad z=0.$

19. If $\frac{6y-4x}{3z-7}=\frac{5z-x}{2y-3z}=\frac{y-2z}{3y-2x}=1$, $\quad x=10,\quad y=7,\quad z=3.$

20. If $x+\frac{1}{2}(y+z)=y+\frac{2}{3}(x+z)=z+\frac{3}{4}(x+y)=x+y+z-4$,

$x=10,\quad y=6,\quad z=2.$

21. If $$\left.\begin{array}{l} \frac{2}{x}-\frac{5}{3y}+\frac{1}{z}=3\frac{4}{27}, \\ \frac{1}{4x}+\frac{1}{y}+\frac{2}{z}=6\frac{11}{72}, \\ \frac{5}{6x}-\frac{1}{y}+\frac{4}{z}=12\frac{1}{36}, \end{array}\right\} \qquad \left\{\begin{array}{l} x=6, \\ y=9, \\ z=\frac{1}{3}. \end{array}\right.$$

22. If $$\left.\begin{array}{l} \frac{xy+86}{3x+2y}=\frac{4x-y}{5}, \\ \frac{xz+4\frac{1}{2}}{x+3z}=\frac{x-z}{2}, \\ \frac{yz+4\frac{1}{7}}{2y+z}=\frac{y+3z}{7}, \end{array}\right\} \qquad \left\{\begin{array}{l} x=6, \\ y=1, \\ z=3. \end{array}\right.$$

23. If $$\left.\begin{array}{l} \frac{3}{x}-\frac{4}{5y}+\frac{1}{z}=7\frac{3}{5}, \\ \frac{1}{3x}+\frac{1}{2y}+\frac{2}{z}=10\frac{1}{6}, \\ \frac{4}{5x}-\frac{1}{2y}+\frac{4}{z}=16\frac{1}{10}, \end{array}\right\} \qquad \left\{\begin{array}{l} x=\frac{1}{2}, \\ y=\frac{1}{3}, \\ z=\frac{1}{4}. \end{array}\right.$$

24. If $$\left.\begin{aligned}&\frac{x+z-3}{14}+\frac{4y-4z+5}{9}=\frac{x-4}{3},\\&\frac{5x-6y+z}{7}+\frac{3x-y+z+1}{11}=\frac{2y+z-5}{3},\\&\frac{5y-8z+4}{12}-\frac{x-2z+5}{6}=2y+3z-2x,\end{aligned}\right\}\qquad\begin{cases}x=25,\\y=16,\\z=\ \ 6.\end{cases}$$

25. If $$\left.\begin{aligned}&x+y+z=0,\\&(a+b)x+(a+c)y+(b+c)z=0,\\&abx+\ acy\ +\ bcz\ =1,\end{aligned}\right\}\qquad\begin{cases}x=\dfrac{1}{(a-c)(b-c)},\\y=\dfrac{-1}{(a-b)(b-c)},\\z=\dfrac{1}{(a-b)(a-c)}.\end{cases}$$

26. If $$\left.\begin{aligned}&x-ay+a^2z-a^3=0,\\&x-by+b^2z-b^3=0,\\&x-cy+c^2z-c^3=0,\end{aligned}\right\}\qquad\begin{cases}x=abc,\\y=ab+ac+bc,\\z=a+b+c.\end{cases}$$

27. If $$\left.\begin{aligned}&5x-11\sqrt{y}+13\sqrt[3]{z}=22,\\&4x+\ 6\sqrt{y}+\ 5\sqrt[3]{z}=31,\\&x\ -\ \sqrt{y}\ +\ \sqrt[3]{z}=\ 2,\end{aligned}\right\}\qquad\begin{cases}x=1,\\y=4,\\z=27.\end{cases}$$

28. If $$\left.\begin{aligned}&x^{-2}y^{-1}z=1\tfrac{1}{2},\\&x^{-1}y\ z^2=18,\\&xy^2\ z^3=108,\end{aligned}\right\}\qquad\begin{cases}x=1,\\y=2,\\z=3.\end{cases}$$

29. If $$\left.\begin{aligned}&x^{-1}y^{-1}z^{-1}=(105)^{-1},\\&x\,y^{-1}z^{-1}=3\times(35)^{-1},\\&x\,y\ \ z^{-1}=2\tfrac{1}{7},\end{aligned}\right\}\qquad\begin{cases}x=3,\\y=5,\\z=7.\end{cases}$$

30. If $$\left.\begin{aligned}&x^2y^3z^4=a,\\&x^3y^4z^2=b,\\&x^4y^2z^3=c,\end{aligned}\right\}\qquad\begin{cases}x=\sqrt[27]{\dfrac{bc^{10}}{a^8}},\\y=\sqrt[27]{\dfrac{ab^{10}}{c^8}},\\z=\sqrt[27]{\dfrac{ca^{10}}{b^8}}.\end{cases}$$

31. If $$\left.\begin{aligned}&xyz=\ 231,\\&xyw=\ 420,\\&yzw=1540,\\&xzw=\ 660,\end{aligned}\right\}\qquad\begin{cases}x=3,\\y=7,\\z=11,\\w=20.\end{cases}$$

32. If $$\left.\begin{aligned} 7x - 2z + 3u &= 17,\\ 4y - 2z + t &= 11,\\ 5y - 3x - 2u &= 8,\\ 4y - 3u + 2t &= 9,\\ 3z + 8u &= 33, \end{aligned}\right\} \qquad \left\{\begin{aligned} x &= 2,\\ y &= 4,\\ z &= 3,\\ u &= 3,\\ t &= 1. \end{aligned}\right.$$

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1. **Shew** that the following equations are not sufficient to determine x, y, z.

$$(1)\quad \left.\begin{aligned} 3x - 2y + 5z &= 14,\\ 2x + y - 8z &= 10,\\ 8x - 3y + 2z &= 38. \end{aligned}\right\} \qquad (2)\quad \left.\begin{aligned} 3x - 2y + 5z &= 14,\\ 6x - 4y - 3z &= 15,\\ 9x - 6y - 7z &= 20. \end{aligned}\right\}$$

$$(3)\quad \left.\begin{aligned} 2z - y + 2z &= 8,\\ 6y + 13 - \frac{x}{2} &= 4x - 1 + 3z,\\ \frac{x}{4} + z - y + \frac{1}{2} &= \frac{1}{4} - \frac{2y}{3} + \frac{5z}{6}. \end{aligned}\right\}$$

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1. If $\dfrac{x+2}{x-1} - \dfrac{4-x}{2x} = 2\frac{1}{3}$, $\qquad x = 3$, or $-\dfrac{4}{5}$.

2. If $125x^2 - 7x = 17\frac{1}{5}$, $\qquad x = \dfrac{2}{5}$, or $-\dfrac{43}{125}$.

3. If $\dfrac{2x(a-x)}{3a-2x} = \dfrac{a}{4}$, $\qquad x = \dfrac{3a}{4}$, or $\dfrac{a}{2}$.

4. If $\dfrac{nx+b}{\sqrt{x}} = \dfrac{na+b}{\sqrt{a}}$, $\qquad x = a$, or $\dfrac{b^2}{n^2a}$.

5. If $\sqrt{a+x} + \sqrt{a-x} = \dfrac{12a}{5\sqrt{a+x}}$, $\qquad x = \dfrac{4a}{5}$, or $\dfrac{3a}{5}$.

6. If $\dfrac{a+x}{\sqrt{a-x}} + \dfrac{a-x}{\sqrt{a+x}} = 2\sqrt{a}$, $\qquad x = \pm a\sqrt{8\sqrt{2}-11}$.

7. If $\dfrac{3\sqrt{x} - x^{\frac{3}{2}}}{x+2} = \dfrac{1\frac{2}{3} + 3\sqrt{x} - 2x}{2\sqrt{x} - 3}$, $\qquad x = 4$, or $\dfrac{1}{25}$.

8. If $ax + 2\sqrt{n^2x + nax^2} = (3x-1) \cdot n$, $\qquad x = \dfrac{n}{n-a}$, or $\dfrac{n}{9n-a}$.

9. If $x + \sqrt{x^2 - ax + b^2} = \dfrac{x^2}{a} + b$, $\qquad x = a$, or $\dfrac{a}{2}\left\{1 \pm \sqrt{5 - \dfrac{8b}{a}}\right\}$.

10. If $a+x=\dfrac{2a\sqrt{1+a^2x}}{a\sqrt{x}+\sqrt{1+a^2x}}$, $x=\dfrac{a}{1+2a\sqrt{a}}$.

11. If $\dfrac{x^2}{8a}+\dfrac{2x}{3}=\sqrt{\dfrac{x^3}{3a}+\dfrac{x^2}{4}-\dfrac{a}{2}}$, $x=6a$, or $-\dfrac{2a}{3}$.

12. If $5\sqrt{62+3x}-\dfrac{1}{2}\sqrt{95\frac{2}{3}-5x}=41$, $x=6\frac{1}{3}$.

13. If $a+x+\sqrt{2ax+x^2}=\dfrac{3x^2-6ax}{4\sqrt{2ax+x^2}}$, $x=\dfrac{8a}{5}$, or $\left(\dfrac{2}{3}\sqrt{3}-1\right)2a$.

14. If $\left(x-\dfrac{ab}{x}\right)^2=\dfrac{a}{2}(a+b)\left(1+\dfrac{a^2}{x^2}\right)$, $x=\sqrt{2ab+a^2}$, or $\sqrt{\dfrac{a}{2}(b-a)}$.

15. If $mqx^2-mnx+pqx-np=0$, $x=\dfrac{n}{q}$, or $-\dfrac{p}{m}$.

16. If $\sqrt{x+\sqrt{2x-1}}-\sqrt{x-\sqrt{2x-1}}=\dfrac{3}{5}\sqrt{\dfrac{10x}{x+\sqrt{2x-1}}}$,

$x=2\frac{1}{2}$, or $\dfrac{5}{8}$.

17. If $(x-a)^2+2\sqrt{x}.(x-a)=a^2+\sqrt{x}$, $x=2a+1\frac{1}{2}\pm\sqrt{2a+1\frac{1}{4}}$.

18. If $x^2(a+x)+bx(a+x)=b-(x+2b)$, $x=-b$, or $\pm\sqrt{\dfrac{a^2}{4}-1}-\dfrac{a}{2}$.

19. If $\sqrt[4]{a+x}+\sqrt[4]{a-x}=b$, $x=\pm\sqrt{a^2-\left(b^2\pm\sqrt{a+\dfrac{b^4}{2}}\right)^4}$.

20. If $\sqrt{2x^2-1}+\sqrt{1-a^2}.\sqrt{1-x^2}=ax$,

$x=\left\{\dfrac{5a^2-6\pm2a\sqrt{2}(a^2-1)}{8a^2-9}\right\}^{\frac{1}{2}}$.

21. If $2x^2-x-2x\sqrt{1-x^2}=1\frac{1}{2}$, $x=\pm\dfrac{1}{2}\sqrt{3}$.

22. If $\dfrac{(1+x)^2}{1+x^3}+\dfrac{(1-x)^2}{1-x^3}=a$, $x=\left\{\dfrac{2}{a}-\dfrac{1}{2}\pm\sqrt{\left(\dfrac{2}{a}\right)^2-\dfrac{3}{4}}\right\}^{\frac{1}{2}}$.

23. If $\dfrac{1+x^3}{(1+x)^3}+\dfrac{1-x^3}{(1-x)^3}=a$, $x=\left\{\dfrac{a+4\mp2\sqrt{3(a+1)}}{a-2}\right\}^{\frac{1}{2}}$.

24. If $16(x^2+2)^{\frac{3}{2}}+\dfrac{3}{\sqrt{x^2+2}}=32x^2+48$, $x=\pm\dfrac{1}{2}$.

25. If $(x+3)^2-2(x^2+3)=2x(x+1)^2$, $x=1$, or -3, or $-\dfrac{1}{2}$.

26. If $\sqrt{1+a}\,.\sqrt{1-x}-\sqrt{1-a}\,.\sqrt{1+x}=2a,$ $x=a\,(3-4a^2),$ or $-a.$

27. If $(1-x)\sqrt{x^2+x^3}-(1+x)\sqrt{x^2-x^3}=\dfrac{\sqrt{\{2+2\sqrt{1-x^2}\}}}{a},$

$$x=\left\{\frac{1}{2}+\frac{1}{a}\pm\sqrt{\frac{1}{4}-\frac{1}{a}}\right\}^{\frac{1}{2}}.$$

28. If $\dfrac{\sqrt{1+a^2}-a\sqrt{1+x^2}}{\sqrt{1+x^2}-x\sqrt{1+a^2}}=a,$ $x=\dfrac{1}{a},$ or $\dfrac{1}{a}\,.\,\dfrac{1-3a^2}{a^2-3}.$

29. If $(a+x)\sqrt{a^2+x^2}=6\,(a-x)^2,$ $x=(9\pm4\sqrt{2})\dfrac{a}{7}.$

30. If $\dfrac{a-\sqrt{2ax-x^2}}{a+\sqrt{2ax-x^2}}=\dfrac{x}{a-x},$ $x=a,$ or $\dfrac{a}{5}.$

31. If $\sqrt{x^2+1}-\sqrt{x^2-1}=\dfrac{x}{\sqrt{x^4-1}},$ $x=\pm\sqrt{\dfrac{2}{3}\sqrt{3}}.$

32. If $n\sqrt{a^2+x^2}+(n-1)\sqrt{x^2-2\,(n-1)\,a^2}=2n-1,$

$$x=\sqrt{(n\pm\overline{n-1}\,.\sqrt{1-a^2})^2-a^2}.$$

33. If $\dfrac{na+n\,.\,\dfrac{n-1}{2}ra}{1+nr}=p,$

$$n=\left(\frac{1}{2}-\frac{1}{r}\right)+\left(\frac{1}{2}+\frac{p}{a}\right)-\frac{1}{2}\pm\sqrt{\left(\frac{1}{2}-\frac{1}{r}\right)^2+\left(\frac{1}{2}+\frac{p}{a}\right)^2-\frac{1}{2^2}}.$$

34. If $x^{2m}-1=a+\dfrac{a}{x^m},$ $x=\sqrt[m]{\dfrac{1}{2}\pm\sqrt{a+\dfrac{1}{4}}}.$

35. If $\sqrt[5]{(a+x)^2}+\sqrt[5]{(a-x)^2}=3\sqrt[5]{a^2-x^2},$ $x=\dfrac{(3\pm\sqrt{5})^5-2^5}{(3\pm\sqrt{5})^5+2^5}a.$

36. If $\dfrac{(27a+8x)^{\frac{2}{3}}}{15x^{\frac{18}{15}}}+\dfrac{8x^{\frac{2}{15}}}{3\sqrt[3]{27a+8x}}=\dfrac{8}{5\sqrt[5]{x}},$ $x=\dfrac{3a}{32}\{3\pm\sqrt{21}\}.$

37. If $a^2b^2x^{\frac{1}{q}}-4\,(ab)^{\frac{3}{2}}\,.\,x^{\frac{p+q}{2pq}}=(a-b)^2\,.\,x^{\frac{1}{p}},$ $x=\left(\dfrac{1}{\sqrt{b}}\pm\dfrac{1}{\sqrt{a}}\right)^{\frac{4pq}{p-q}}.$

38. If $\sqrt[2pq]{x^{p+q}}-\dfrac{1}{2}\,.\,\dfrac{a^2-b^2}{a^2+b^2}\,.\,(\sqrt[p]{x}+\sqrt[q]{x})=0,$ $x=\left(\dfrac{a\pm b}{a\mp b}\right)^{\frac{2pq}{q-p}}.$

39. If $(a^{4m}+1)(x^{\frac{1}{2}}-1)^2=2\,(x+1),$ $x=\left\{\dfrac{a^{2m}\pm1}{a^{2m}\mp1}\right\}^2.$

40. If $x-3=\dfrac{3+4\sqrt{x}}{x},$ $x=\dfrac{7\pm\sqrt{13}}{2}.$

41. If $x + 7\sqrt[3]{x} = 22,$ $\quad x = 8, \text{ or } \{-1 \pm \sqrt{-10}\}^3.$

42. If $x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4,$ $\quad x = 1, \text{ or } \frac{-3 \pm \sqrt{5}}{2}.$

43. If $x^6 - 7x^3 = 8,$ $\quad x = 2, \text{ or } -1.$

44. If $\sqrt{x} - \frac{8}{x} = \frac{7}{\sqrt{x} - 2},$ $\quad x = 1, \text{ or } 16.$

45. If $x^2(x^2 - 23) - 10x(x^2 - 24) = 649,$ $\quad x = \frac{1}{2}\{5 \pm 3\sqrt{29}\}.$

46. If $2x^{\frac{3}{2}}(x^3 + a^3)^{\frac{1}{2}} = 2x^2(x + 2a) + a^2(x - a),$ $\quad x = \frac{a}{2}, \text{ or } -a.$

47. If $x^2 - 7 = \sqrt{\{x^2 - 42x + 89\}},$ $\quad x = 2, \text{ or } -5.$

48. If $\frac{x}{2} + \frac{63}{\sqrt[4]{x}} = \frac{220\frac{1}{2}}{\sqrt{x}} + 49\sqrt{x} - 1196,$ $\quad x = 2401, \text{ or } \frac{1}{16}\{-7 \pm \sqrt{37}\}^4.$

49. If $\frac{1 + x^3}{(1 + x)^3} = a,$ $\quad x = \frac{1 + 2a \pm \sqrt{12a - 3}}{2(1 - a)}.$

50. If $\frac{1 + x^5}{(1 + x)^5} = a,$

$$x = \frac{\sqrt{4a + 1}(\sqrt{5} + \sqrt{4a + 1}) + [10(6a - 1) + 2\sqrt{5}(4a + 1)^{\frac{3}{2}}]^{\frac{1}{2}}}{4(1 - a)}.$$

51. If $\sqrt{x^2 + \sqrt{x}} = x - 1 + 3\sqrt{x},$ $\quad x = \frac{1}{4}, \text{ or } \frac{31 \pm 5\sqrt{37}}{18}.$

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1. If $\left.\begin{array}{l} x - y = a, \\ \text{and } y^2 + ay + bx = 0, \end{array}\right\}$ $\quad \left\{\begin{array}{l} x = a - b, \text{ or } 0, \\ y = -b, \text{ or } -a. \end{array}\right.$

2. If $\left.\begin{array}{r} 2x^2 + 3xy + y^2 = 20, \\ \text{and } 5x^2 + 4y^2 = 41, \end{array}\right\}$ $\quad \left\{\begin{array}{l} x = 1, \text{ or } \frac{13}{\sqrt{21}}, \\ y = 3, \text{ or } \frac{2}{\sqrt{21}}. \end{array}\right.$

3. If $\left.\begin{array}{l} \sqrt[3]{x} + \sqrt[3]{y} = 3, \\ \text{and } x + y = 9, \end{array}\right\}$ $\quad \left\{\begin{array}{l} x = 8, \text{ or } 1, \\ y = 1, \text{ or } 8. \end{array}\right.$

4. If $\left.\begin{array}{l} x - y = 8, \\ \text{and } x^4 - y^4 = 14560, \end{array}\right\}$ $\quad \left\{\begin{array}{l} x = 11, \\ y = 3. \end{array}\right.$

5. If $\left.\begin{array}{l} x + y = 5, \\ \text{and } (x^2 + y^2) \cdot (x^3 + y^3) = 455, \end{array}\right\}$ $\quad \left\{\begin{array}{l} x = 3, \\ y = 2. \end{array}\right.$

6. If $\left.\begin{array}{l}\sqrt{\dfrac{x}{y}}+\sqrt{\dfrac{y}{x}}=\dfrac{61}{\sqrt{xy}}+1, \\ \text{and } \sqrt[4]{x^3y}+\sqrt[4]{xy^3}=78,\end{array}\right\}$ $\left\{\begin{array}{l}x=81, \\ y=16.\end{array}\right.$

7. If $\left.\begin{array}{l}x^4+y^4=641, \\ \text{and } x^3y+y^3x=290,\end{array}\right\}$ $\left\{\begin{array}{l}x=5, \\ y=2.\end{array}\right.$

8. If $\left.\begin{array}{l}x^2+xy+y^2=37, \\ x^2+xz+z^2=28, \\ y^2+yz+z^2=19,\end{array}\right\}$ $\left\{\begin{array}{l}x=4, \\ y=3, \\ z=2.\end{array}\right.$

9. If $\left.\begin{array}{l}\dfrac{5\sqrt{x+y}}{x}+\dfrac{5\sqrt{x+y}}{y}=10\frac{2}{3}, \\ \text{and } \dfrac{3\sqrt{x-y}}{y}-\dfrac{3\sqrt{x-y}}{x}=\dfrac{4}{5},\end{array}\right\}$ $\left\{\begin{array}{l}x=2\frac{1}{2}, \\ y=1\frac{1}{2}.\end{array}\right.$

10. If $\left.\begin{array}{r}xz=y^2, \\ (x+y)(z-x-y)=3, \\ (x+y+z)(z-x-y)=7,\end{array}\right\}$ $\left\{\begin{array}{l}x=1, \text{ or } 9, \\ y=2, \text{ or } -6, \\ z=4.\end{array}\right.$

11. If $\left.\begin{array}{l}x+y+z=216, \\ \sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}=12, \\ \sqrt[3]{y^2}+\sqrt[3]{z^2}=\sqrt[3]{x^2}+\sqrt[3]{yz}-12,\end{array}\right\}$ $\left\{\begin{array}{l}x=125, \\ y=64, \\ z=27.\end{array}\right.$

12. If $\left.\begin{array}{r}(x-2)^2+(y-3)^2+(z-1)^2=24, \\ xy+xz+yz=63, \\ 2x+3y+z=30,\end{array}\right\}$ $\left\{\begin{array}{l}x=6, \\ y=5, \\ z=3.\end{array}\right.$

13. If $\left.\begin{array}{l}x^3+y^3+xy(x+y)=13, \\ \text{and } (x^2+y^2)x^2y^2=468,\end{array}\right\}$ $\left\{\begin{array}{l}x=3, \text{ or } -2, \\ y=-2, \text{ or } 3.\end{array}\right.$

14. If $\left.\begin{array}{l}\dfrac{y}{x}-\dfrac{x}{x+y}=\dfrac{x^2-y^2}{y}, \\ \text{and } \dfrac{x}{y}-\dfrac{x+y}{x}=\dfrac{y}{x},\end{array}\right\}$ $\left\{\begin{array}{l}x=-\dfrac{1}{9}, \\ y=-\dfrac{1}{18}.\end{array}\right.$

15. If $\left.\begin{array}{l}\dfrac{x}{y}-\dfrac{y}{x}=\dfrac{x+y}{x^2+y^2}, \\ \text{and } \dfrac{x^2}{y^2}-\dfrac{y^2}{x^2}=\dfrac{x-y}{y^2},\end{array}\right\}$ $\left\{\begin{array}{l}x=\dfrac{1}{4}\{1\pm\sqrt{2}\}, \\ y=\dfrac{1}{4}.\end{array}\right.$

16. If $\left.\begin{array}{l}\dfrac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}=x,\\ \text{and } \dfrac{x}{y}=\sqrt{\dfrac{1+x}{1-y}},\end{array}\right\}$ $\left\{\begin{array}{l}x=\dfrac{1}{4},\\ y=\dfrac{1}{5}.\end{array}\right.$

17. If $\left.\begin{array}{l}x^2-6\sqrt{x^2y}=27,\\ \text{and } x-2\sqrt{xy}=3,\end{array}\right\}$ $\left\{\begin{array}{l}x=9,\\ y=1.\end{array}\right.$

18. If $\left.\begin{array}{l}(x+y)^2=x^4+x^2y^2+y^4,\\ \text{and } x^4+4y^4=4xy(2y^2-x^2),\end{array}\right\}$ $\left\{\begin{array}{l}x=\sqrt{\dfrac{8\pm2\sqrt{3}}{13}},\\ y=\sqrt{\dfrac{11\pm6\sqrt{3}}{13}}.\end{array}\right.$

19. If $\left.\begin{array}{l}x+\sqrt{x^2-y^2}=\dfrac{8}{y}(\sqrt{x+y}+\sqrt{x-y}),\\ \text{and } (x+y)^{\frac{3}{2}}-(x-y)^{\frac{3}{2}}=26,\end{array}\right\}$ $\left\{\begin{array}{l}x=5,\\ y=4.\end{array}\right.$

20. If $\left.\begin{array}{l}x+\sqrt{x^2-y^2}=\dfrac{a}{y}(\sqrt{x+y}+\sqrt{x-y}),\\ \text{and } \sqrt[4]{x+y}+\sqrt[4]{x-y}=y,\end{array}\right\}$

$$x=\frac{(1+a^2)^{\frac{2}{3}}}{2\sqrt[3]{2}}\left\{1+\left(\frac{2a}{1+a^2}\right)^2\right\},\text{ and } y=a\sqrt[3]{\frac{4}{1+a^2}}.$$

21. If $\left.\begin{array}{l}x^2-y^2=17ay+3y^2+4x\sqrt{2ay-y^2},\\ \text{and } x^2+y^2=2ay+8\sqrt{y}\,.\,(x\sqrt{a}-y\sqrt{y}),\end{array}\right\}$ $\left\{\begin{array}{l}x=7a,\\ y=a.\end{array}\right.$

22. If $\left.\begin{array}{l}x-\sqrt{x^2-y^2}=\dfrac{a}{y}(\sqrt{x+y}+\sqrt{x-y}),\\ \text{and } (x+y)^{\frac{3}{2}}-(x-y)^{\frac{3}{2}}=b,\end{array}\right\}$. $\left\{\begin{array}{l}x=\dfrac{b+2a}{3\sqrt[3]{4a}},\\ y=\sqrt[6]{4a}\,.\sqrt{\dfrac{b-a}{3}}.\end{array}\right.$

23. If $\left.\begin{array}{l}\dfrac{x+y-\sqrt{x^2+y^2}}{x+y+\sqrt{x^2+y^2}}=\dfrac{2x}{a},\\ \text{and } \dfrac{x}{y}=\sqrt{\dfrac{a+x}{a-y}},\end{array}\right\}$ $\left\{\begin{array}{l}x=\dfrac{-11\pm\sqrt{153}}{16}\,.\,a,\\ y=\dfrac{13\mp\sqrt{153}}{8}\,.\,a.\end{array}\right.$

24. If $\left.\begin{array}{l}\sqrt{x+y}-\sqrt{x-y}=a,\\ \text{and } \sqrt[4]{x+y}+\sqrt[4]{x-y}=b,\end{array}\right\}$ $\left\{\begin{array}{l}x=\dfrac{a^2}{4}+\left\{\dfrac{1}{4}\left(b^2+\dfrac{a^2}{b^2}\right)\right\}^2,\\ y=\dfrac{a}{4}\left(b^2+\dfrac{a^2}{b^2}\right).\end{array}\right.$

25. $$\text{If } \left.\begin{array}{l}\sqrt{x}-\sqrt{y}=\sqrt{y+2},\\ \text{and } \quad x+8=8\sqrt{y+2},\end{array}\right\} \qquad \begin{cases}x=4,\\ y=\dfrac{1}{4}.\end{cases}$$

26. $$\text{If } \left.\begin{array}{l}x^4+y^4=1+2xy+3x^2y^2,\\ \text{and } x^3+y^3=2y^2x+2y^2+x+1,\end{array}\right\} \qquad \begin{cases}x=2,\\ y=1.\end{cases}$$

27. $$\text{If } \left.\begin{array}{l}x^2+y^2=3xy,\\ \text{and } x^5+y^5=2,\end{array}\right\} \qquad \begin{cases}x=\dfrac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}-1),\\ y=\dfrac{\sqrt[5]{10}}{2\sqrt{5}}(\sqrt{5}+1).\end{cases}$$

28. $$\text{If } \left.\begin{array}{l}x^2-xy=6,\\ \text{and } x^2+y^2=61,\end{array}\right\} \qquad \begin{cases}x=6,\\ y=5.\end{cases}$$

29. $$\text{If } \left.\begin{array}{l}(\sqrt{x})^{\sqrt[4]{x}+\sqrt[4]{y}}=(\sqrt{y})^{\frac{8}{3}},\\ \text{and } (\sqrt{y})^{\sqrt[4]{x}+\sqrt[4]{y}}=(\sqrt{x})^{\frac{2}{3}},\end{array}\right\} \qquad \begin{cases}x=\left\{-\dfrac{1}{2}\pm\dfrac{1}{6}\sqrt{57}\right\}^8,\\ x=\left\{-\dfrac{1}{2}\pm\dfrac{1}{6}\sqrt{57}\right\}^4.\end{cases}$$

30. $$\text{If } \left.\begin{array}{l}5ax+12y(a-x)=0,\\ \text{and } x^2-y^2+a^2=0,\end{array}\right\} \qquad \begin{cases}x=\dfrac{4a}{3}, \quad \text{or} \quad \dfrac{3a}{4},\\ y=\dfrac{5a}{3}, \quad \text{or} \quad -\dfrac{5a}{4}.\end{cases}$$

31. $$\text{If } \left.\begin{array}{l}x^4y^2-4=4xy^2-\dfrac{y^6}{4},\\ \text{and } \quad x^3-3=xy(x-y),\end{array}\right\} \qquad \begin{cases}x=1,\\ y=2.\end{cases}$$

32. $$\text{If } \left.\begin{array}{r}\sqrt{x^2+\sqrt[3]{x^4y^2}}+\sqrt{y^2+\sqrt[3]{x^2y^4}}=a,\\ \text{and } x+y+3\sqrt[3]{bxy}=b,\end{array}\right\} \qquad \begin{cases}x=\dfrac{1}{8}\{b^{\frac{1}{3}}+\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}}\}^3,\\ y=\dfrac{1}{8}\{b^{\frac{1}{3}}-\sqrt{2a^{\frac{2}{3}}-b^{\frac{2}{3}}}\}^3.\end{cases}$$

33. $$\text{If } \left.\begin{array}{l}(x^2+y^2)\cdot\dfrac{y}{x}=8\tfrac{2}{3},\\ \text{and } (x^2-y^2)\cdot\dfrac{x}{y}=7\tfrac{1}{2},\end{array}\right\} \qquad \begin{cases}x=3,\\ y=2.\end{cases}$$

34. $$\text{If } \left.\begin{array}{l}(x^2+y^2)(x+y)=15xy,\\ \text{and } (x^4+y^4)(x^2+y^2)=85x^2y^2,\end{array}\right\} \qquad \begin{cases}x=2, \ \text{or } 4,\\ y=4, \ \text{or } 2.\end{cases}$$

35. $$\text{If } \left.\begin{array}{l}(x^2-xy+y^2)\cdot(x^2+y^2)=91,\\ \text{and } (x^2-xy+y^2)\cdot(x^2+xy+y^2)=133,\end{array}\right\} \qquad \begin{cases}x=2, \ \text{or} -3,\\ y=3, \ \text{or} -2.\end{cases}$$

36. If $(x-2)y-\sqrt{xy}\,.\,(y^2-1)=2y^2-x,$ and $\dfrac{xy}{4}=\dfrac{\sqrt{xy}-12}{xy-18},$ $\left\{\begin{array}{l} x=8,\ \text{or}-12, \\ y=2,\ \text{or}-\dfrac{4}{3}. \end{array}\right.$

$$\left.\begin{array}{l} \text{If } (x-2)y-\sqrt{xy}\,.\,(y^2-1)=2y^2-x, \\ \text{and } \dfrac{xy}{4}=\dfrac{\sqrt{xy}-12}{xy-18}, \end{array}\right\} \qquad \left\{\begin{array}{l} x=8,\ \text{or}-12, \\ y=2,\ \text{or}-\dfrac{4}{3}. \end{array}\right.$$

37. $$\left.\begin{array}{l} \text{If } 5-2\sqrt{y+2}=\dfrac{9x^2}{64}-(\sqrt{x}-3\sqrt{y})^2, \\ \text{and } \dfrac{7}{y}-10\sqrt{\dfrac{x}{y}}=x-16, \end{array}\right\} \qquad \left\{\begin{array}{l} x=4, \\ y=\dfrac{1}{4}. \end{array}\right.$$

38. $$\left.\begin{array}{l} \text{If } \sqrt{x+y}+\sqrt{x-y}=\dfrac{xy-y\sqrt{x^2-y^2}}{a}, \\ \text{and } \sqrt[4]{x+y}+\sqrt[4]{x-y}=b, \end{array}\right\}$$

$$x=\frac{1}{4}\left\{\frac{b^2}{2}+\frac{\sqrt[3]{2a^2}}{b^2}\right\}^2+2\sqrt[3]{2a^2},\ \text{and}\ y=\frac{b^2}{2}\sqrt{\frac{a}{2}+\frac{a}{b^2}}.$$

39. $$\left.\begin{array}{l} \text{If } (b+x)x+ay=(x+y)^2-2\sqrt{xy}\,.\,\sqrt{a-x}\,.\,\sqrt{b-y}, \\ \text{and } \sqrt{y}+\sqrt{a-x}=\sqrt{\dfrac{nx}{y}}, \end{array}\right\}$$

$$x=a+\frac{n-b}{b^2}\{n-b\pm\sqrt{ab\,(b-1)+n\,(n-b)}\}^2,\ \text{and}\ y=b-n.$$

40. $$\left.\begin{array}{l} \text{If } x^2+y^2=x+\dfrac{a}{2}, \\ \text{and } x^3(x-2)+6y^2x(x-1)+y^4=\dfrac{b}{2}-x, \end{array}\right\}$$

$$\left\{\begin{array}{l} x=\dfrac{1}{2}\pm\dfrac{1}{4}\sqrt{4a+2\mp2\sqrt{8a^2-4a-8b+1}}, \\ y=\pm\dfrac{1}{4}\sqrt{4a+2\pm2\sqrt{8a^2-4a-8b+1}}. \end{array}\right.$$

41. $$\left.\begin{array}{l} \text{If } xy+\dfrac{61}{2}x^{\frac{1}{3}}y^{\frac{2}{3}}-410\left(\dfrac{y}{x}\right)^{\frac{1}{3}}=\dfrac{1200}{x}, \\ \text{and } 3x^{\frac{4}{3}}+16y^{\frac{4}{3}}+42(xy)^{\frac{2}{3}}=4(xy)^{\frac{1}{3}}\{5x^{\frac{2}{3}}+11y^{\frac{2}{3}}\}, \end{array}\right\} \qquad x=8,\ y=27.$$

42. $$\left.\begin{array}{l} \text{If } \dfrac{x^{\frac{1}{3}}}{y^{\frac{1}{3}}}(x^{\frac{2}{3}}-1)+\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(2x^{\frac{2}{3}}-1)=\dfrac{4y^{\frac{1}{3}}}{x^{\frac{1}{3}}}(y^{\frac{1}{3}}+x^{\frac{1}{3}})+\dfrac{3y}{x^{\frac{1}{3}}}+2, \\ \text{and } \dfrac{x^{\frac{4}{3}}}{y^{\frac{4}{3}}}-\dfrac{2x^{\frac{2}{3}}}{y}-\dfrac{2x^{\frac{1}{3}}}{y^{\frac{1}{3}}}=\dfrac{133}{36}\,.\,\dfrac{1}{y^{\frac{2}{3}}}-\dfrac{2}{x^{\frac{1}{3}}}-\dfrac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}. \end{array}\right\}$$

$$\left\{\begin{array}{l} x=27,\ \text{or}\ -8, \\ y=\ \ 8,\ \text{or}\ -27. \end{array}\right.$$

43. If $\left.\begin{array}{l} x^4 = mx + ny, \\ \text{and } y^4 = my + nx, \end{array}\right\}$ $x = y = \sqrt[3]{m+n}$,

$$\text{or } x = \frac{1}{2}\sqrt[3]{4m + 2n(a \mp \sqrt{a^2 - 4})}, \text{ and } y = \sqrt[3]{m + \frac{1}{2}n\,(a \pm \sqrt{a^2 - 4})},$$

$$\text{where } a = \frac{-(m+n) \pm \sqrt{m^2 - 2mn + 5n^2}}{2n}.$$

44. If $\left.\begin{array}{l} 30\sqrt{\dfrac{x^{\frac{2}{3}} + y^{\frac{2}{3}}}{x^{\frac{4}{3}}y^{\frac{4}{3}}}} + 40\sqrt{\dfrac{x^{\frac{4}{3}}y^{\frac{4}{3}}}{x^{\frac{2}{3}} + y^{\frac{2}{3}}}} = 241, \\ \text{and } \left\{1 + \left(\dfrac{y}{x}\right)^{\frac{2}{3}}\right\}.\left\{3x^{\frac{4}{3}}y^{\frac{2}{3}} + \dfrac{91}{216}\sqrt{x^2 + x^{\frac{4}{3}}y^{\frac{2}{3}}}\right\} = \left(\dfrac{5}{6}\right)^3 - x^2 - y^2, \end{array}\right\}$

$$x = \pm\frac{8}{27}, \text{ or } \pm\frac{1}{8}; \text{ and } y = \pm\frac{1}{8}, \text{ or } \pm\frac{8}{27}.$$

45. If $\left.\begin{array}{l} (1 - x^2)^2.(1 + y^2) - (1 + x^2)^2.(1 - y^2) = 4x^2\sqrt{1 + y^4}, \\ \text{and } 4xy = \sqrt{2}\,(1 - x^2)(1 - y^2), \end{array}\right\}$

$$x = \frac{(\mp\sqrt{2\sqrt{3} - 2})\sqrt[4]{2\sqrt{3}} \pm 2}{\sqrt{2\sqrt{3}} \pm (\sqrt{3} + 1)}, \text{ and } y = \pm\sqrt{\frac{\sqrt{3} - 1}{\sqrt{2\sqrt{3}}}}.$$

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1. The trinomial $ax^2 + bx + c$ becomes 42 when $x = 4$, 22 when $x = 3$, and 8 when $x = 2$; what are the values of a, b, c?

Ans. $a = 3$; $b = -1$; and $c = -2$.

2. Solve the equation $x^2 - 5x - 24 = 0$, without completing the square. Ans. $x = 8$, or -3.

3. Prove that in simple equations of two unknown quantities there is only one pair of values of the unknown quantities which will satisfy the two equations.

4. If x_1, x_2, represent the two values of x which satisfy the equation $ax^2 + bx + c = 0$, prove that $\dfrac{x_1}{x_2} + \dfrac{x_2}{x_1} = \dfrac{b^2 - 2ac}{ac}$; and verify this formula by the equation $2x^2 - 7x + 3 = 0$.

5. If α, β, represent the two roots of the equation

$$x^2 - (1 + a)x + \frac{1}{2}(1 + a + a^2) = 0,$$

shew that $\alpha^2 + \beta^2 = a$.

6. Shew that $x + \frac{1}{x}$ cannot be less than 2, whatever positive quantity be substituted for x.

7. If a proposed equation be reduced to the form $ax = ax + c$, what conclusion is to be drawn as to the value of x? Ans. $x = \infty$.

8. Shew that there can be no more than *three* distinct values of x which satisfy the equation $ax^3 + bx^2 + cx + d = 0$, and that their sum $= \frac{-b}{a}$.

9. Is $2\{(x - a)(x - b) + (a - x)(a - b) + (b - x)(b - a)\} = (a - b)^2 + (x - a)^2 + (x - b)^2$ an 'Identity' or an 'Equation'? Ans. The former.

10. Given $x + y + z = \frac{14}{3}x = \frac{7}{2}y$, find the value of $\frac{x + y + z}{z}$. Ans. 2.

11. Given $x - y = 7z$, and $x - z = 4y$, find the value of $\frac{y - z}{x}$. Ans. $\frac{1}{9}$.

12. Given $x^2 + y^2 = 123z$, and $x^2 - y^2 = 27z$, find the value of $\frac{xy}{z}$.

Ans. 60.

13. Given that $ab - \frac{1}{2}(a + b)(p + q) + pq = 0$,

and $cd - \frac{1}{2}(c + d)(p + q) + pq = 0$;

shew that $\frac{(p - q)^2}{2} = \frac{(a - c)(a - d)(b - c)(b - d)}{(a + b - c - d)^2}$.

14. If the same value of x satisfies both the equations $ax^2 + bx + c = 0$, and $a'x^2 + b'x + c' = 0$, what is the relation subsisting between a, b, c, a', b', c'? Ans. $(ac' - a'c)^2 = (ab' - a'b)(bc' - b'c)$.

15. Find the relation subsisting between the coefficients in the equations $a_1x + b_1y = c_1$, $a_2x + b_2y = c_2$, $a_3x + b_3y = c_3$, that they may be satisfied by the same values of x and y.

Ans. $(a_1b_2 - a_2b_1)c_3 = (a_1c_2 - a_2c_1)b_3 + (c_1b_2 - c_2b_1)a_3$.

16. Find the relations subsisting between the coefficients, when the equations

$$ax + by + cz = a'x + b'y + c'z = a''x + b''y + c''z = 1,$$

are equivalent to no more than two distinct equations. And shew that in this case the values of x, y, z, assume the form $0 \div 0$.

Ans. $(a'b - ab')(a''c' - a'c'') = (a'c - ac')(a''b' - a'b'')$.

17. Eliminate b and c from the equations, $a + b + c = -p$, $ab + ac + bc = q$, $abc = -r$. Ans. $a^3 + pa^2 + qa + r = 0$.

18. Find the value of $ax^2 + by^2 + cz^2$, when $ax^3 = by^3 = cz^3$, and $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{d}$. Ans. $(a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}})^3 . d^2$.

19. Eliminate *a, b, c,* from the equations

$$\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m+\left(\frac{z}{c}\right)^m=1=\left(\frac{a}{d}\right)^n+\left(\frac{b}{d}\right)^n+\left(\frac{c}{d}\right)^n, \text{ and } \frac{x^m}{a^{m+n}}=\frac{y^m}{b^{m+n}}=\frac{z^m}{c^{m+n}},$$

Ans. $x^{\frac{mn}{m+n}}+y^{\frac{mn}{m+n}}+z^{\frac{mn}{m+n}}=d^{\frac{mn}{m+n}}$.

PROBLEMS.

1. *A* is twice as old as *B*. Twenty-two years ago he was four times as old as *B*. What is *A*'s age? Ans. 66 years.

2. Nine years ago *A* was three times as old as *B*, but now he is only twice as old. Required the respective ages of *A* and *B*.
Ans. *A* is 36, *B* is 18.

3. The ages of a father and his son are 80 years; and if the age of the son be doubled, it will exceed the father's age by 10 years. Find the age of each. Ans. 50 and 30.

4. A certain party is composed of three times as many men as women; and when four men have left together with their wives, there remain *four times* as many men as women. How many were there of each sex at first? Ans. 36 men, 12 women.

5. Divide the number 90 into two such parts, that if half of the greater part be added to the double of the smaller, the result is the original number 90. Ans. 60, and 30.

6. Find two numbers in the ratio of 4 to 5, such that if 6 be added to the greater number, and 1 to the smaller, the square roots of the resulting numbers shall differ by 1. Ans. 24, and 30; or 8, and 10.

7. There is a certain number of which the cube root is one-fifth of the square root: find it. Ans. 15625.

8. There is a certain fraction, which, if 1 be added to its numerator, becomes $\frac{1}{3}$; but if 1 be added to its denominator, it becomes $\frac{1}{4}$. What is the fraction? Ans. $\frac{4}{15}$.

9. There is a certain fraction, which, if 1 be taken from its denominator, and added to the numerator, becomes $\frac{4}{5}$; but, if 1 be taken from its numerator and added to the denominator, becomes $\frac{1}{2}$. What is the fraction? Ans. $\frac{7}{11}$.

10. A person distributed *p* shillings among *n* persons, giving 9*d*. to some, and 15*d*. to the rest. How many were there of each?
Ans. $\frac{1}{2}(5n-4p)$ at 9*d*., and $\frac{1}{2}(4p-3n)$ at 15*d*.

11. A labourer is engaged for n days, on condition that he receives p pence for every day he works, and pays q pence for every day he is idle. At the end of the time he receives a pence. How many days did he work, and how many was he idle?

Ans. He worked $\frac{nq+a}{p+q}$, and was idle $\frac{np-a}{p+q}$, days.

12. A person, being asked what o'clock it was, answered that it was between 5 and 6, and that the hour and minute hands were together. Required the time of day. Ans. 27m. $16\frac{4}{11}$s. past 5.

13. Find the time after h o'clock at which the hour and minute hands of a watch are distant d of the minute divisions from each other.

Ans. $\frac{12}{11}(5h \pm d)$.

14. There are two places 154 miles distant from each other, from which two persons A and B set out at the same instant with a design to meet on the road, A travelling at the rate of 3 miles in 2 hours, and B at the rate of 5 miles in 4 hours. How long and how far did each travel before they met? Ans. A travelled 84 miles, and 56 hours,
B .... 70 ....

15. Find a number, such, that whether it is divided into *two* or *three* equal parts, the continued product of the parts shall be the same.

Ans. $6\frac{3}{4}$.

16. A person bought a certain number of sheep for £94: having lost 7 of them, he sold one-fourth of the remainder at prime cost for £20. How many sheep had he at first? Ans. 47.

17. A farmer buys m sheep for p£, and sells n of them at a gain of £5 per cent.: how must he sell the remainder that he may clear 10 per cent. on the whole?

Ans. $\frac{22m-21n}{20m(m-n)}\cdot p$£ each.

18. A boy at a fair spends his money in oranges. If he had received 5 more for his money, they would have cost a halfpenny each less; but if 5 less, a halfpenny each more. How much did he spend?

Ans. 2*s.* 6*d.*

19. A hare is 80 of its own leaps before a greyhound, and takes 3 leaps for every 2 taken by the greyhound, but the latter passes over as much ground in one leap as the former does in two. How many leaps will the hare have taken before it is caught? Ans. 240.

20. A courier passing through a certain place (P) travels at the rate of 5 miles in 2 hours. Four hours afterwards another passes through the same place travelling the same way at the rate of 7 miles in two hours. How far from the place (P) is the first overtaken by the second?

Ans. 35 miles.

21. Two travellers, A and B, set out from two places P and Q at the same time; A from P intending to pass through Q, and B from Q intending to travel the same way. After A had overtaken B, and they had computed their travels, it was found that the distance A had travelled together with the distance B had travelled made up 30 miles; that A had passed through Q 4 hours before; and that B at his rate of travelling was 9 hours journey distant from P. Required the distance between the two places P and Q. Ans. 6 miles.

22. The rent of a farm is paid in certain fixed numbers of quarters of wheat and barley: when wheat is at 55*s.* and barley 33*s.* per quarter, the portions of rent by wheat and barley are equal to one anther; but when wheat is at 65*s.* and barley at 41*s.* per quarter, the rent is increased by £7. What is the corn-rent?

Ans. 6 qrs. of wheat; 10 qrs. of barley.

23. A constable in pursuit of a thief at a uniform pace finds by inquiry that the thief is travelling $1\frac{1}{2}$ miles per hour quicker than himself; he therefore doubles his speed after the first 4 hours, and takes the thief at the end of 6 hours and 20 minutes from the time of his starting. Given that the thief had a start of 1 hour, and never varied his speed, find the rates of travelling of the two parties, and the distance at which the capture took place.

Ans. Constable's speed at first $8\frac{1}{4}$ miles per hour,
Thief's speed throughout $9\frac{3}{4}$ ....
Required distance $71\frac{1}{2}$ miles.

24. At an election where each elector may give two votes to different candidates, but only one to the same, it is found on casting up the poll, that of the candidates A, B, C, A had 158 votes, B had 132, C had 58. Now 26 voted for A only, 30 for B only, and 28 for C only. How many voted for A and B jointly; how many for A and C; and how many for B and C? Ans. For A and B, 102; A and C, 30; B and C, 0.

25. During a panic there was a run on two bankers, A and B; B stopped payment at the end of three days, in consequence of which the alarm increased, and the daily demand for cash on A being tripled, A failed at the end of two more days. Now if A and B had joined their capitals together, they might both have stood the run as it was at first for 7 days, at the end of which time B would have been indebted to A £4000. What was the daily demand for cash on A at the beginning of the run? Ans. £2000.

26. There is a waggon with a mechanical contrivance by which the difference of the number of revolutions of the wheels on a journey is noted. The circumference of the fore-wheel is a feet, and of the hind-wheel b feet; what is the distance gone over, when the fore-wheel has made n revolutions more than the hind-wheel? Ans. $\dfrac{abn}{b-a}$ feet.

27. A person has two casks containing a certain quantity of wine in each. He wishes to have an equal quantity in each; and in order to have this, he pours out of the first cask into the second as much as the second contained at first; then he pours from the second into the first as much as was left in the first; and then again from the first into the second as much as was left in the second. At last there are exactly a gallons in each cask. How many gallons were in each cask at first?

Ans. $\frac{11a}{8}$, and $\frac{5a}{8}$.

28. A person rows from Cambridge to Ely (a distance of 20 miles) and back again in ten hours, the stream flowing uniformly in the same direction all the time; and he finds that he can row 2 miles against the stream in the same time that he rows 3 miles with it. Find the velocity of the stream, and the times of going and returning.

(1) Ans. $\frac{5}{6}$ of a mile per hour. (2) Ans. 4 hours. (3) Ans. 6 hours.

29. The Gas company engage to light a shop, for six days in a week, with 5 large and 3 small burners, but having by them only one large burner, they supply the deficiency with 5 small ones. The shopkeeper, not finding this light sufficient, procures two small burners more, and at the same time agrees for the lights to burn double the usual time on Saturday nights, for which additional gas he was to pay £1. 11*s*. How much did he pay a year altogether? Ans. £5. 5*s*.

30. A shopkeeper, on account of bad book-keeping, knows neither the weight nor the prime cost of a certain article which he had purchased. He only recollects, that if he had sold the whole at 30*s*. per lb., he would have gained £5 by it, and if he had sold it at 22*s*. per lb., he would have lost £15 by it. What was the weight and prime cost of the article?

Ans. Weight 50lbs. Cost 28*s*. per lb.

31. A book is so printed, that each page contains a certain number of lines, and each line a certain number of letters. If we wished each page to contain 3 lines more, and each line 4 letters more, then there would be 224 letters more than before in a page; but if we wished to have 2 lines less in each page, and 3 letters less in each line, then the page would contain 145 letters less than at first. How many lines are there in each page, and how many letters in each line? Ans. 29 lines, and 32 letters.

32. When wax candles are half-a-crown a pound, a composition is invented of such a nature, that a candle made of it will burn two-thirds of the time in which a wax candle of the same thickness and one-fourth as heavy again will continue burning. Supposing the two candles give an equally bright light, what must be charged per lb. for the composition that it may be as *cheap* as wax? Ans. 2*s*. 1*d*.

33. *A*, *B*, *C*, *D*, *E* play together on the condition that he who loses shall give to all the rest as much as they have already. First *A* loses, then *B*, then *C*, then *D*, then *E*. All lose in turn, and yet at the end of the

fifth game they all have the same sum, viz. £32. How much had each before they began to play?

Ans. *A* £81, *B* £41, *C* £21, *D* £11, *E* £6.

34. A man at his death leaves property to the amount of £5850 to be divided among three sons, four daughters, and his widow, in manner following:—viz. the share of two sons is to be equal to that of three daughters, and the mother's share half that of a son and a daughter taken together. Find each person's share.

Ans. Each son £900, daughter £600, mother £750.

35. *A* and *B* engaged to reap equal quantities of wheat, and *A* began half an hour before *B*. They stopped at 12 o'clock, and rested an hour, observing that just half the whole work was done. *B*'s part was finished at 7 o'clock, and *A*'s at a quarter before 10. Supposing them to have laboured uniformly, determine the times at which they commenced.

Ans. *A* at $\frac{1}{2}$ past 4, *B* at 5 o'clock.

36. A cistern can be filled by three different pipes; by the 1st in $1\frac{1}{3}$ hours, by the 2nd in $3\frac{1}{3}$ hours, and by the third in 5 hours. In what time will this cistern be filled when all three pipes are opened at once?

Ans. 48 minutes.

37. In a tithe-commutation the rent-charge was apportioned at 3*s.* an acre, and in the 1st year the rates payable on the rent-charge wanted £6 of 10 per cent. on the whole receipts. The next year the rates were doubled, and amounted to 15 per cent. on the receipts: what was the number of acres?

Ans. 1600.

38. *A* and *B* drink from a cask of beer for 2 hours, after which *A* falls asleep, and *B* drinks the remainder in 2 hours and 48 minutes: but if *B* had fallen asleep, and *A* had continued to drink, it would have taken him 4 hours and 40 minutes to finish the cask. In what time would each singly be able to drink the whole?

Ans. *A* in 10 hours, *B* in 6 hours.

39. There is a number composed of two figures, of which the figure in the units' place is triple of that in the tens', and if 36 be added to the number the sum is expressed by the same digits reversed. What is the number?

Ans. 26.

40. Find a number composed of two digits, which is equal to four times the sum of the digits, and such that, by reversing the digits, a number is formed equal to seven times that sum.

Ans. 48.

41. The mail train upon a railway starts a certain time after a luggage-train from the same terminus, and the time is so adjusted that, before arriving at the other terminus, the trains will exactly escape collision and no more. It happens, however, that from an accident to the engine the speed of the luggage-train is suddenly reduced one-half after performing two-thirds of its journey, and a collision takes place a miles from the end of it. The proper speeds of the trains being m and n miles

per hour, $(m > n)$ find the length of the railway, and the difference of times of starting.
(1) Ans. $3\left(2-\frac{n}{m}\right)a$. (2) Ans. $3 . \frac{m-n}{mn}\left(2-\frac{n}{m}\right)a$.

42. Three persons divide a certain sum of money amongst them in the following manner:—A takes the n^{th} part of the whole together with $\frac{a}{n}$ £; B takes the n^{th} part of the remainder together with $\frac{a}{n}$ £; C takes the n^{th} part of what now remains together with $\frac{a}{n}$ £; and then nothing remains. Find the sum.
Ans. $\frac{3n^2-3n+1}{(n-1)^3} . a$ £.

43. A pack of p cards is distributed into n heaps, so that the number of pips on the lowest cards, together with the number of cards laid upon them, is the same number m for each heap, and the number of cards remaining is found to be r; required the number of pips on all the lowest cards.
Ans. $(m+1)n+r-p$.

44. If a men or b boys can dig m acres in n days, find the number of boys whose assistance will be required to enable $\overline{a-p}$ men to dig $\overline{m+p}$ acres in $\overline{n-p}$ days.
Ans. $\frac{pb}{a}\left(1+\frac{m+n}{n-p}.\frac{a}{m}\right)$.

45. Supposing the sum of 51 cards in a common pack to be $10n+a$, (where $a < 10$), prove the value of the last card to be $10-a$, the court-cards reckoning for 10, and the others for as much as is the number of pips upon each. Find also the value of n.
Ans. $n = 33$.

46. If a oxen in m weeks eat b acres of grass, and c oxen eat d acres in n weeks, how many oxen will eat e acres in p weeks, supposing the grass to grow uniformly.
Ans. $\frac{m-p}{m-n}.\frac{nce}{pd}-\frac{n-p}{m-n}.\frac{mae}{pb}$.

47. The distance between two places is a, and on the first day $\frac{1}{m}$th of the journey from one to the other is performed; on the 2nd day $\frac{1}{n}$th of the remainder; then $\frac{1}{m}$th and $\frac{1}{n}$th of the remainders alternately on succeeding days. Find the distance gone over in $2p$ days.
Ans. $a\left\{1-\left(1-\frac{1}{m}\right)^p.\left(1-\frac{1}{n}\right)^p\right\}$.

48. To complete a certain work A requires m times as long a time as B and C together; B requires n times as long as A and C together; and C requires p times as long as A and B together. Compare the times in which each would do it, and prove that

$$\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1.$$

49. S_1, S_2, $S_3, \ldots S_{n+1}$ are $\overline{n+1}$ stones placed in a straight line a yard from each other, and X is another assumed station in the same line produced; two persons set out from S_1, the one to carry the stones separately to S_1, and the other to X; find the distance from S_{n+1} to X, that the latter may travel exactly twice as far as the former. Ans. $\frac{n(n+2)}{2n+1}$.

50. Two clocks are striking the hour together, and are heard to strike 19 times. There is a difference of two seconds in their time, and one strikes every three, the other every four, seconds. What is the hour they strike? it being observed that, when the clocks strike in the same second, the sounds cannot be distinguished, so as to determine whether one or both strike in that second, and that this is the case with the last stroke of the faster clock. Ans. 11 o'clock.

51. The hold of a vessel partly full of water (which is uniformly increased by a leak) is furnished with two pumps worked by A and B, of whom A takes three strokes to two of B's, but four of B's throw out as much water as five of A's. Now B works for the time in which A alone would have emptied the hold. A then pumps out the remainder, and the hold is cleared in 13 hrs. 20 min. Had they worked together, the hold would have been emptied in 3 hrs. 45 min., and A would have pumped out 100 gallons more than he did. Required the quantity of water in the hold at first, and the horary influx at the leak.

Ans. Quantity in the hold 1200 gallons.
Horary influx 120 gallons.

INEQUALITIES.

1. If $4x-7<2x+3$, and $3x+1>13-x$, find x. Ans. $x=4$.

2. What is the integral value of x, when $\frac{1}{4}(x+2)+\frac{1}{3}x<\frac{1}{2}(x-4)+3$, and $>\frac{1}{2}(x+1)+\frac{1}{3}$? Ans. $x=5$.

3. Which is greater $x-y$ or $(\sqrt{x}-\sqrt{y})^2$?
Ans. The former, if $x>y$.

4. Shew that $\frac{a}{b^2}+\frac{b}{a^2}>\frac{1}{a}+\frac{1}{b}$.

5. Shew that $\sqrt{a^2-b^2}+\sqrt{a^2-(a-b)^2}>a$, if $a>b$.

6. If $x^2=a^2+b^2$, and $y^2=c^2+d^2$, which is greater, xy or $ac+bd$?
Ans. xy.

7. Which is greater, n^3+1 or n^2+n? Ans. n^3+1, unless $n=1$.

8. Which is greater, $3(1 + a^2 + a^4)$, or $(1 + a + a^2)^2$?
Ans. The former, unless $a = 1$.

9. Which is greater, $2(1 + a^2 + a^4)$, or $3(a + a^3)$?
Ans. The former, unless $a = 1$.

10. Shew that $\dfrac{n^2 - n + 1}{n^2 + n + 1}$ lies between 3 and $\dfrac{1}{3}$ for all real values of n.

11. Shew that $abc > (a + b - c)(a + c - b)(b + c - a)$, unless $a = b = c$.

12. Shew that $abc > (2a - b)(2b - c)(2c - a)$, unless $a = b = c$.

13. Shew that $ab(a + b) + ac(a + c) + bc(b + c)$ is between $6abc$, and $2(a^3 + b^3 + c^3)$, a, b, c being positive quantities.

14. Shew that $(a+b+c)^3 > 27abc$, and $< 9(a^3+b^3+c^3)$, unless $a = b = c$.

15. If $a < x$, shew that $(x + a)^3 - x^3 < 7ax^2$.

16. Shew that $\sqrt[3]{n} > \sqrt[4]{n+1}$, for all values of n not less than 3.

17. Shew that $(a^2 + b^2 + 1)(c^2 + d^2 + 1) > (ac + bd + 1)^2$, unless $a = c$, and $b = d$.

18. Shew that $\dfrac{1}{3}(a + b + c) > \sqrt[3]{abc}$, unless $a = b = c$.

19. The double of a certain number increased by 7 is not greater than 19, and its triple diminished by 5 is not less than 13. What is the number?
Ans. 6.

RATIOS, PROPORTION, AND VARIATION.

1. Which is greater, $3 : 5$, or $5 : 8$? Ans. The latter.

2. Prove that $a : b$ is a greater ratio than $ax : bx + y$, and a less atio than $ax : bx - y$.

3. Prove that $a^3 + b^3 : a^2 + b^2 > a^2 + b^2 : a + b$.

4. Find the ratio *compounded* of $a : x$, $x : y$, and $y : b$. Ans. $a : b$.

5. Find the ratio compounded of $x + a : x + b$, and $a(x + b) : b(x + a)$.
Ans. $a : b$.

6. What quantity must be added to each of the terms of the ratio : b, that it may become the ratio $c : d$?
Ans. $\dfrac{ad - bc}{c - d}$.

7. Prove that, if $a : b$ is a greater ratio than $c : d$, $a + c : b + d$ is a less ratio than $a : b$ but a greater than $c : d$.

8. Four given numbers are represented by a, b, c, d; required the quantity which added to each will make them proportionals.

Ans. $\dfrac{bc - ad}{a - b - c + d}$.

9. If four numbers be proportionals, shew that there is no number which, being added to each, will leave the resulting four numbers proportionals.

10. If $a : b = c : d$, shew that $\dfrac{2a + 3b}{4a + 5b} = \dfrac{2c + 3d}{4c + 5d}$.

11. If four quantities of the same kind be proportionals, prove that the greatest and least together are greater than the other two together.

12. If $(a+b)^2 : (a-b)^2 :: b + c : b - c$, shew that $a : b :: \sqrt{2a - c} : \sqrt{c}$.

13. If $x : y :: a^3 : b^3$, and $a : b :: \sqrt[3]{c + x} : \sqrt[3]{d + y}$, shew that $cy = dx$.

14. If $a : b :: c : d$, shew that $a(a + b + c + d) = (a + b)(a + c)$.

15. If $a : b :: c : d$, shew that $\sqrt{a - b} : \sqrt{c - d} :: \sqrt{a} - \sqrt{b} : \sqrt{c} - \sqrt{d}$ $:: \sqrt{a} + \sqrt{b} : \sqrt{c} + \sqrt{d}$.

16. If $a : b :: c : d$, shew that

$$\frac{1}{ma} + \frac{1}{nb} + \frac{1}{pc} + \frac{1}{qd} = \frac{1}{bc}\left\{\frac{a}{q} + \frac{b}{p} + \frac{c}{n} + \frac{d}{m}\right\}.$$

17. If $a_1 : b_1 :: a_2 : b_2 :: a_3 : b_3 ::$ &c., shew that

$$(a_1^2 + a_2^2 + a_3^2 + \text{\&c.})(b_1^2 + b_2^2 + b_3^2 + \text{\&c.}) = (a_1b_1 + a_2b_2 + a_3b_3 + \text{\&c.})^2.$$

18. If the difference between a and b be small when compared with either of them, shew that the ratio $\sqrt[m]{a} - \sqrt[m]{b} : \sqrt[n]{a} - \sqrt[n]{b}$ is nearly equal to $n\sqrt[m]{a} : m\sqrt[n]{a}$.

19. If $\dfrac{\sqrt[n]{x} - m\sqrt[n]{y}}{\sqrt[n]{x} + m\sqrt[n]{y}} = \dfrac{\sqrt[2n]{x} - m\sqrt[2n]{x - y}}{\sqrt[2n]{x} + m\sqrt[2n]{x - y}}$, shew that $x : y :: 1 \pm \sqrt{5} : 2$.

20. Find the number to which if 1 and 3 be successively added, the resulting numbers are in the proportion $2 : 7$.

Ans. $-\dfrac{1}{5}$.

21. Find two numbers in the ratio $3 : 4$, and of which the sum : the sum of their squares $:: 7 : 50$. Ans. 6, and 8.

22. If m shillings in a row reach as far as n sovereigns, and a pile of p shillings is as high as a pile of q sovereigns, compare the values of equal bulks of gold and silver. Ans. Val. of gold : val. of silver $:: 20n^2q : m^2p$.

23. Given that $y \propto x$, and when $x = 2$, $y = 10$, required the equation between x and y. Ans. $y = 5x$.

24. Given that $y^2 \propto a^2 - x^2$; and when $x = \sqrt{a^2 - b^2}$, $y = \frac{b^2}{a}$; find the equation between x and y. Ans. $y = \frac{b}{a}\sqrt{a^2 - x^2}$.

25. Given that $s \propto t^2$, when f is constant; and $s \propto f$, when t is constant: also $2s = f$, when $t = 1$. Find the equation between f, s, and t. Ans. $s = \frac{1}{2}ft^2$.

26. Given that y is equal to the sum of two quantities, one of which varies as x, and the other varies inversely as x^2; and when $x = 1, 2$, $y = 6, 5$, respectively. Find the equation between x and y. Ans. $y = 2x + \frac{4}{x^2}$.

27. Given that y is equal to the sum of three quantities, the 1st of which is invariable, the 2nd varies as x, and the 3rd varies as x^2. Also when $x = 1, 2, 3$, $y = 6, 11, 18$, respectively. Express y in terms of x. Ans. $y = 3 + 2x + x^2$.

28. Given that $x \propto \frac{1}{y^m}$, and $y \propto \frac{1}{z^n}$; also when $x = a$, $z = c$; find the equation between x and z. Ans. $az^{mn} = c^{mn}x$.

29. If $y = r + s$, whilst $r \propto x$, and $s \propto \sqrt{x}$; and if, when $x = 4$, $y = 5$, and when $x = 9$, $y = 10$; shew that $6y = 5(x + \sqrt{x})$.

30. If $a + b \propto a - b$, prove that $a^2 + b^2 \propto ab$; and if $a \propto b$, prove that $a^2 - b^2 \propto ab$.

31. If $\frac{x}{y} \propto x + y$, and $\frac{y}{x} \propto x - y$, shew that $x^2 - y^2$ is invariable.

32. If $ax + by = cx + dy$, prove that $x \propto y$.

33. If $a \propto b$, and $b \propto c$, shew that $(a^2 + b^2)^{\frac{3}{2}} \propto c^3$.

34. If $A \propto B$, and $B \propto C$, shew that

$$mA + nB + pC \propto m'\sqrt{AB} + n'\sqrt{AC} + p'\sqrt{BC}.$$

35. If $A \propto B$, and $B \propto \sqrt{AC}$, and $C \propto \sqrt[3]{A^2B} + \sqrt[3]{AB^2}$, shew that

$$m\sqrt[4]{AB} - n\sqrt[4]{BC} \propto p\sqrt{A} + q\sqrt{B}.$$

36. If $mA + nB \propto pA - qB$, and $m'A - n'C \propto p'B + q'C$; shew that

$$(\alpha_1\sqrt{A} - \beta_1\sqrt{B} + \gamma_1\sqrt{C})^2 \propto \alpha_2 A + \beta_2 B + \gamma_2 C.$$

37. A sphere of metal is known to have a hollow space about its centre in the form of a concentric sphere, and its weight is $\frac{7}{8}$ of the

weight of a solid sphere of the same substance and radius; compare the inner and outer radii, having given that the weights of spheres of the same substance $\propto$ (radii)^3. Ans. 1 : 2.

38. A locomotive engine without a train can go 24 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of waggons attached. With four waggons its speed is 20 miles an hour. Find the greatest number of waggons which the engine can move. Ans. 143.

39. The value of diamonds $\propto$ the square of their weight, and the square of the value of rubies $\propto$ the cube of their weight. A diamond of a carats is worth m times the value of a ruby of b carats, and both together are worth c£. Required the values of a diamond and of a ruby, each weighing n carats.

Ans. Value of diamond $= \dfrac{mcn^2}{(m+1)a^2}$; value of ruby $= \dfrac{cn^{\frac{3}{2}}}{(m+1)b^{\frac{3}{2}}}$.

40. If the prices of two trees containing p and q cubic feet of timber be a£ and b£; required the price of a tree containing r cubic feet, supposing the values of the timber and bark to be respectively proportional to the m^{th} and n^{th} powers of the quantity of timber in the tree.

Ans. $\dfrac{(bp^n - aq^n)\,r^m + (aq^m - bp^m)\,r^n}{p^n q^m - p^m q^n}$.

ARITHMETICAL PROGRESSION.

1. Find the 64th term of the series 4, $6\frac{1}{2}$, 9, &c. Ans. $161\frac{1}{2}$.

2. Find the 30th term of the series -27, -20, -13, &c. Ans. 176.

3. Find the sum of 50 terms in A. P. of which the first is 3, and the last 199. Ans. 5050.

4. Find the 7th term, and the sum of 7 terms, of the series $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, &c.

(1) Ans. $-\frac{1}{2}$, (2) Ans. 0.

5. Find the 6th term, and the sum of 6 terms, of the series $\frac{2}{3}, \frac{7}{15}, \frac{4}{15}$, &c. (1) Ans. $-\frac{1}{3}$, (2) Ans. 1.

6. Find the n^{th} term of $1 + 3 + 5 + 7 + \ldots\ldots\ldots\ldots$ Ans. $2n-1$.

and of $2 + 2\frac{1}{3} + 2\frac{2}{3} + \ldots\ldots\ldots\ldots$ Ans. $\frac{1}{3}(n+5)$.

and of $13 + 12\frac{2}{3} + 12\frac{1}{3} + \ldots\ldots\ldots$ Ans. $\frac{1}{3}(40-n)$.

7. Sum the following series:

(1). $1+2+3+4+\ldots$ to n terms. Ans. $n\frac{n+1}{2}$.

(2). $1+3+5+7+\ldots$ to n terms. Ans. n^2.

(3). $2+2\frac{1}{3}+2\frac{2}{3}+\ldots\ldots$ to n terms. Ans. $\frac{n}{6}(n+11)$.

(4). $13+12\frac{2}{3}+12\frac{1}{3}+\ldots$ to n terms. Ans. $\frac{n}{6}(79-n)$.

(5). $\frac{1}{2}-\frac{2}{3}-\frac{11}{6}-\ldots\ldots$ to n terms. Ans. $\frac{n}{12}(13-7n)$.

(6). $\frac{1}{4}+\frac{3}{8}+\frac{1}{2}+\ldots\ldots\ldots$ to n terms. Ans. $\frac{n}{16}(n+3)$.

(7). $\frac{5}{7}+1+1\frac{2}{7}+\ldots\ldots\ldots$ to n terms. Ans. $\frac{n}{7}(n+4)$.

(8). $\frac{1}{3}+\frac{5}{6}+\frac{4}{3}+\ldots\ldots\ldots$ to n terms. Ans. $\frac{n}{12}(3n+1)$.

(9). $-9-7-5-$ &c. to 20 terms. Ans. 200.

(10). $\frac{5}{7}+\frac{2}{9}+$ &c. to 101 terms. Ans. $-2412\frac{7}{9}$.

(11). $2\frac{1}{3}+3\frac{5}{24}+4\frac{1}{12}+$ &c. to 24 terms. Ans. $297\frac{1}{2}$.

(12). $\frac{a-b}{a+b}+\frac{3a-2b}{a+b}+\ldots$ to n terms. Ans. $\frac{n}{a+b}\left\{na-\frac{n+1}{2}b\right\}$.

(13). $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+\ldots\ldots\ldots$ Ans. $\frac{n-1}{2}$.

(14). $\left(\frac{1}{a}-\frac{n}{x}\right)+\left(\frac{1}{a}-\frac{n-1}{x}\right)+\left(\frac{1}{a}-\frac{n-2}{x}\right)+\ldots$ to n terms.

Ans. $\frac{n}{a}-n\,.\,\frac{n+1}{2}\,.\,\frac{1}{x}$.

(15). $(s+a)+(s+a+d)+(s+a+2d)+\ldots$ to n terms, where $=(2a+\overline{n-1}\,.\,d)\frac{n}{2}$. Ans. $(n+1)s$.

8. Given the first term (a) of a series in Arithmetical Progression, he common difference (b), and (s) the sum of the series to n terms; find n.

Ans. $n=\frac{1}{2}-\frac{a}{b}\pm\sqrt{\frac{2s}{b}+\left(\frac{1}{2}-\frac{a}{b}\right)^2}$.

9. How many terms of the series 19, 18, 17, &c. amount to 124?
Ans. 8, or 31.

10. The sum of a series in Arithmetical Progression is 72, the first term 17, and the common difference -2; find the number of terms and explain the double answer. Ans. 6, or 12.

11. Given $s=40$, $a=7$, and $b=2$. Find n. Ans. $n=4$, or -10.

12. In any Arith^c. Prog^n. of which a is the first term, and $2a$ the common difference, prove that the number of terms, which must be taken to make a sum s, is $\sqrt{\frac{s}{a}}$, s being assumed such that $\frac{s}{a}$ is *any* square number, but no other.

13. Insert 40 Arith^c. Means between 0 and 20.
Ans. $\frac{20}{41}$, $\frac{40}{41}$, $1\frac{19}{41}$, $1\frac{39}{41}$, &c. ...... $19\frac{21}{41}$.

14. Insert 15 Arith^c. Means between 3 and 47.
Ans. $5\frac{3}{4}$, $8\frac{1}{2}$, $11\frac{1}{4}$, &c. .... $44\frac{1}{4}$.

15. If the Arith^c. Mean between a and b be twice as great as the Geom^c. Mean, shew that $a : b :: 2+\sqrt{3} : 2-\sqrt{3}$.

16. If the Arith^c. Mean between a and b be m times as great as the Harmonic Mean, shew that $a : b :: \sqrt{m}+\sqrt{m-1} : \sqrt{m}-\sqrt{m-1}$.

17. Shew that if the same number of Arith^c. Means be inserted between every two terms of an A. P. the whole will be in A. P.

18. Find the series in A. P. having 29 terms of which the first is 3 and the last 17. Ans. 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$, &c. ... 17.

19. Find the series in A. P. of $\overline{n+7}$ terms of which the sum of the first n terms is 40, the sum of the next 4 is 86, and that of the last 3 is 96.
Ans. 2, 5, 8, 11, 14, 17, 20, 23, 26, 29, 32, 35.

20. The first two terms of an A. P. are $189\frac{1}{3}$, and $103\frac{3}{4}$; and the sum of all the terms is $-147\frac{3}{4}$; what is the last term, and what the number of terms? (1) Ans. $-238\frac{7}{12}$. (2) Ans. 6.

21. Divide $\frac{n}{7}(n+4)$ into n parts, such that each part shall exceed the one immediately preceding by a fixed quantity. Ans. $\frac{5}{7}$, 1, $1\frac{2}{7}$, &c.

22. The n^{th} term of an Arith^c. Prog^n. is $\frac{1}{6}(3n-1)$, prove that the sum of n terms is $\frac{n}{12}(3n+1)$, and find the series. Ans. $\frac{1}{3}$, $\frac{5}{6}$, $\frac{4}{3}$, $\frac{11}{6}$, &c.

23. There are two series in Arith^{c}. Prog^{n}., the sums of which to n terms are as $13-7n : 3n+1$; prove that their first terms are as $3 : 2$, and their second terms as $-4 : 5$.

24. The $\overline{n+1}^{\text{th}}$ term of a series in Arith^{c}. Prog^{n}. is $\dfrac{ma-nb}{a-b}$, required the sum of the series to $\overline{2n+1}$ terms. Ans. $\dfrac{ma-nb}{a-b}\cdot(2n+1)$.

25. Shew that the sum of the $\overline{m-n}^{\text{th}}$ and $\overline{m+n}^{\text{th}}$ terms of any Arith^{c}. Prog^{n}. is equal to twice the m^{th} term.

26. Determine the relation between a, b, and c, that they may be the p^{th}, q^{th}, and r^{th} terms of an Arithmetical Progression.

Ans. $(q-r)a+(r-p)b+(p-q)c=0$.

27. In an Arithmetic Progression, if the $\overline{p+q}|^{\text{th}}$ term $=m$, and the $\overline{p-q}|^{\text{th}}$ term $=n$, shew that the p^{th} term $=\dfrac{1}{2}(m+n)$, and the q^{th} term $=m-(m-n)\dfrac{p}{2q}$.

28. If the m^{th} term of an Arithmetic Progression $=n$, and the n^{th} term $=m$, of how many terms will the sum be $\dfrac{1}{2}(m+n)(m+n-1)$, and what will be the last of them?

(1) Ans. $m+n$, or $m+n-1$. (2) Ans. 0, or 1.

29. In an Arithmetic Progression a is the first term, b the common difference, and S_n the sum of n terms, prove that

$$S_n+S_{n+1}+S_{n+2}+\ldots \text{ to } n \text{ terms} = (3n-1)\frac{na}{2}+(7n-2)(n-1)\frac{nb}{6}.$$

30. $S_1, S_2, S_3, \ldots\ldots S_p$ are the sums of p Arithmetic Progressions to n terms; the first terms are 1, 2, 3, &c. and the common differences 1, 3, 5, 7, &c. Shew that $S_1+S_2+S_3+\ldots+S_p=(np+1)\dfrac{np}{2}$.

31. Prove the following forms in Arithmetical Progression:

$$a=\frac{b}{2}\pm\sqrt{\left(l+\frac{b}{2}\right)^2-2bs},$$

$$b=2\left\{\frac{l}{n-1}-\frac{s}{n(n-1)}\right\}; \quad b=\frac{(l+a)(l-a)}{2s-(l+a)};$$

$$s=\frac{l+a}{2b}(l-a+b); \quad n=\frac{l-a}{b}+1.$$

32. In the two series 2, 5, 8, &c. and 3, 7, 11, &c. each continued to 100 terms, find how many terms are identical. Ans. 25.

GEOMETRICAL AND HARMONICAL PROGRESSION.

1. FIND the 12th term of the series 30, 15, $7\frac{1}{2}$, &c. Ans. $\frac{15}{1024}$.

2. The first two terms of a series in Geom^{c}. Prog^{n}. are $\frac{3}{7}$ and $\frac{8}{9}$; find the Common Ratio, and the third term. (1) Ans. $2\frac{2}{27}$ (2) Ans. $1\frac{205}{243}$.

3. Find the Common Ratio and the fifth term of $3\frac{3}{8}$, $2\frac{1}{4}$, $1\frac{1}{2}$, &c.

(1) Ans. $\frac{2}{3}$ (2) Ans. $\frac{2}{3}$.

4. Sum the following series:

(1) $1-2+4-8+\ldots$ to n terms. Ans. $\frac{1}{3}\{1-(-2)^{n}\}$.

(2) $\frac{1}{5}-\frac{2}{15}+\frac{4}{45}-\ldots$ to n terms. Ans. $\frac{3}{5^{2}}\left\{1-\left(-\frac{2}{3}\right)^{n}\right\}$.

(3) $1-\frac{1}{4}+\frac{1}{16}-\ldots$ to n terms. Ans. $\frac{4}{5}\left\{1-\left(-\frac{1}{4}\right)^{n}\right\}$.

(4) $\frac{1}{3}+\frac{1}{2}+\frac{3}{4}+\ldots$ to n terms. Ans. $\frac{2}{3}\left\{\frac{3^{n}}{2^{n}}-1\right\}$.

(5) $\frac{5}{3}+1+\frac{3}{5}+\ldots$ *in inf.* Ans. $4\frac{1}{6}$.

(6) $\frac{2}{3}-\frac{1}{2}+\frac{3}{8}+\ldots$ *in inf.* Ans. $\frac{8}{21}$.

(7) $\frac{\sqrt{2}+1}{\sqrt{2}-1}+\frac{1}{2-\sqrt{2}}+\frac{1}{2}+\ldots$ *in inf.* Ans. $4+3\sqrt{2}$.

(8) $a^{p}+a^{p+q}+a^{p+2q}+\ldots$ to n terms. Ans. $a^{p}\cdot\frac{a^{nq}-1}{a^{q}-1}$.

(9) $x-y+\frac{y^{2}}{x}-\frac{y^{3}}{x^{2}}+\ldots$ to n terms. Ans $\frac{x^{2}}{x+y}\cdot\left\{1-\left(-\frac{y}{x}\right)^{n}\right\}$.

(10) $\frac{a}{x}\sqrt{\frac{3}{2}}+\sqrt{\frac{a}{x}}+\sqrt{\frac{2}{3}}+\ldots\ldots$ Ans. $\frac{1}{6^{\frac{1}{2}}\cdot(3a)^{\frac{n-3}{2}}x}\cdot\frac{(2x)^{\frac{n}{2}}-(3a)^{\frac{n}{2}}}{(2x)^{\frac{1}{2}}-(3a)^{\frac{1}{2}}}$.

5. Of each of the following series find the nth term, and the sum of n terms:—

(1) $1+5+13+29+61+\ldots$ (1) Ans. $2^{n+1}-3$. (2) Ans. $4(2^{n}-1)-3n$.

(2) $2+6+14+30+\ldots$ (1) Ans. $2^{n+1}-2$. (2) Ans. $4(2^{n}-1)-2n$.

(3) $1+3+7+15+31+\ldots$ (1) Ans. $2^{n}-1$. (2) Ans. $2(2^{n}-1)-n$.

(4) $3+6+11+20+\ldots$ (1) Ans. 2^n+n. (2) Ans. $2^{n+1}+\frac{1}{2}\{n^2+n-4\}$.

(5) $\frac{1}{1}+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\ldots$ (1) Ans. $\frac{2^n-1}{2^{n-1}}$. (2) Ans. $2(n-1)+\frac{1}{2^{n-1}}$.

(6) $2+4+\frac{14}{3}+\frac{44}{9}+\frac{134}{27}+\ldots$ (1) Ans. $5-\frac{1}{3^{n-2}}$. (2) Ans. $5n-\frac{9}{2}\left(1-\frac{1}{3^n}\right)$.

6. $(s-a)+(s-\overline{a+ar})+(s-\overline{a+ar+ar^2})+\ldots$ to n terms, and to ∞, where $s=a\cdot\frac{r^n-1}{r-1}$. (1) Ans. $\frac{nar^n}{r-1}-\frac{ar}{(r-1)^2}(r^n-1)$. (2) Ans. $\frac{ar}{(1-r)^2}$.

7. Find the sum of $(1^3-1)+(2^3-2)+(3^3-3)+$ &c. to n terms.

Ans. $\frac{1}{4}(n+2)(n^3-n)$.

8. Given $x-1+2(x-2)+3(x-3)+$ &c. to 6 terms $=14$; find x.

Ans. $x=5$.

9. The first term of a Geometric Series, continued to ∞, is 1, and any term is equal to the sum of all the succeeding terms; find the series.

Ans. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\ldots$

10. Insert three Geometric Means between $\frac{1}{9}$ and 9. Ans. $\frac{1}{3}$, 1, 3.

11. Insert seven Geometric Means between 2 and 13122.

Ans. 6, 18, 54, 162, 486, 1458, 4374.

12. Insert two Geometric Means between $\frac{16}{27}$ and 2. Ans. $\frac{8}{9}$, $\frac{4}{3}$.

13. The sum of an infinite Geometric Series is 3, and the sum of its first two terms is $2\frac{2}{3}$; find the series.

Ans. $2+\frac{2}{3}+\frac{2}{9}+\ldots$, or $4-\frac{4}{3}+\frac{4}{9}-\ldots\ldots$

14. Prove the following forms in Geometric Progression:—

$$l(s-l)^{n-1}-a(s-a)^{n-1}=0, \qquad r=\sqrt[n-1]{\frac{l}{a}},$$

$$r=\frac{s-a}{s-l}, \qquad s=\frac{\sqrt[n-1]{l^n}-\sqrt[n-1]{a^n}}{\sqrt[n-1]{l}-\sqrt[n-1]{a}}.$$

15. In a Geometric Progression, if the $\overline{p+q}^{\text{th}}$ term $=m$, and the $\overline{p-q}^{\text{th}}$ term $=n$, shew that the p^{th} term $=\sqrt{mn}$, and the q^{th} term $=m\left(\frac{n}{m}\right)^{\frac{p}{2q}}$. Also, if P be the p^{th} term, and Q the q^{th} term, shew that the n^{th} term $=\left(\frac{P^{n-q}}{Q^{n-p}}\right)^{\frac{1}{p-q}}$.

16. Find the relation between a, b, and c, that they may be the p^{th}, q^{th}, and r^{th} terms of a Geometric Series. Ans. $a^{q-r}b^{r-p}c^{p-q}=1$.

17. Required the sum of the first p terms of the series whose n^{th} term is $na+a^n$.

Ans. $\frac{1}{2}p(p+1)a+a\cdot\frac{a^p-1}{a-1}$.

18. Given $\frac{2}{3}$, $\frac{5}{7}$, the first two terms of an A. P., find the sum of 15 terms; and if the same quantities be the first two terms of a G. P., find the sum of 15 terms.

(1) Ans. 15. (2) Ans. $\frac{28}{3}\left\{\left.\overline{\frac{15}{14}}\right|^{15}-1\right\}$.

19. If a, b, c, d are quantities in G. P., prove that $a^2+b^2+c^2>(a-b+c)^2$; and that $(a+b+c+d)^2=(a+b)^2+(c+d)^2+2(b+c)^2$.

20. If there be any number of quantities in G. P., r the common ratio, and S_m the sum of the first m terms, prove that the sum of the products of every two $=\frac{r}{r+1}\cdot S_m\cdot S_{m-1}$.

21. Prove that in any G. P. the sum of the first and last terms is greater than the sum of any other two terms taken equidistant from the beginning and end of the series.

22. The first two terms of a series in Harmonical Progression are a, b; continue the series to three more terms.

Ans. $a, b, \frac{ab}{2a-b}, \frac{ab}{3a-2b}, \frac{ab}{4a-3b}$.

23. Given a and b the first two terms of an Harmonic Progression, find the n^{th} term.

Ans. $\frac{ab}{(n-1)a-(n-2)b}$.

24 Insert two Harmonic Means between 6 and 24. Ans. 8, 12.

25. Insert six Harmonic Means between 3 and $\frac{6}{23}$.

Ans. $1\frac{1}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$.

26. If a, b, c be three terms in Harmonical Progression, shew that $a^2+c^2>2b^2$.

27. The sum of three consecutive terms in Harmonical Progression is $1\frac{1}{12}$, and the first term is $\frac{1}{2}$; find the series. Ans. $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c.

28. Shew that the Arith$^{\text{c}}$., Geom$^{\text{c}}$., and Harmonic Means between any two quantities, are themselves in Geom$^{\text{c}}$. Prog$^{\text{n}}$.

29. There are four quantities, of which the 1st three are in Arith^c^., and the last three in Harm^c^. Prog^n^., prove that the 1st : 2nd :: 3rd : 4th.

30. Find the relation between a, b, and c, that they may be the p^{th}, q^{th}, and r^{th} terms of an Harm^c^. Prog^n^. Ans. $(p-q)ab+(r-p)ac+(q-r)bc=0$.

31. If a, b, c be in G. P., shew that $\log_a n$, $\log_b n$, $\log_c n$, are in H. P.

PERMUTATIONS AND COMBINATIONS.

1. THE number of Permutations of n things taken four together = six times the number taken three together; find n. Ans. $n=9$.

2. The number of Permutations of 15 things taken r together = ten times the number taken $\overline{r-1}$ together; find r. Ans. $r=6$.

3. How many days can 5 persons be placed in different positions about a table at dinner? Ans. 120.

4. How many different sums may be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling, and a sixpence? Ans. 63.

5. In the Permutations formed out of a, b, c, d, e, f, g, taken all together, how many begin with ab? How many with abc? How many with $abcd$? (1) Ans. 120. (2) Ans. 24. (3) Ans. 6.

6. Of the Combinations of 10 letters, a, b, c, &c. taken 5 together, in how many will a occur? Ans. 126.

7. How many different Permutations can be formed from the letters of the word '*Algebra*' taken all together? Ans. 2520.

How many from the letters of '*Proposition*'? Ans. 1663200.

8. At an election, where every voter may vote for any number of candidates not greater than the number to be elected, there are 4 candidates, and 3 members to be chosen; in how many ways may a man vote? Ans. 14.

9. From a company of 50 police-men 4 are taken every night to guard the police-station. On how many different nights can a different selection be made; and on how many of these will any particular man be engaged? (1) Ans. 230300. (2) Ans. 18424.

10. How many combinations are there of 52 things taken 13 together; that is, how many different hands may a person hold at the game of whist? Ans. 635013559600.

11. Find the number of different triangles into which a polygon of n sides may be divided by joining the angular points.

Ans. $\frac{1}{6}n(n-1)(n-2)$.

12. Prove that the total number of different combinations of n things taken 1, 2, 3, ... n at a time is $2^n - 1$.

13. The total number of combinations of $2n$ things = 65 × the total number of combinations of n things; find n. Ans. $n = 6$.

14. Shew that the total number of combinations that can be formed out of $n + 1$ things is more than twice the number that can be formed out of n things.

15. If there be any unknown number of beans in a bag, prove that the chance of bringing out an odd number taken at random is greater than that of bringing out an even number, excluding the case of bringing out none at all.

16. In how many ways can 8 persons be seated at a round table, so that all shall not have the same neighbours in any two arrangements? Ans. 5040.

17. Out of 17 consonants and 5 vowels, how many words can be made having two consonants and one vowel in each? Ans. 4080.

18. Find the number of words which can be formed by taking the 24 letters of the alphabet, 6 at a time, each word containing two vowels. Ans. 27907200.

19. Find the number of combinations that can be formed out of the letters of the word '*Notation*' taken three together. Ans. 22.

20. If there be two dice, one of which has n and the other $\overline{n + r}$ faces, each die being marked in the usual manner from 1 upwards, find the number of different throws which can be made with them.

Ans. $\frac{1}{2} n(n + 1) + nr$.

21. Find the number of Permutations with repetitions (that is, allowing quantities which recur to be combined as if they were different) of n things taken r together. Ans. n^r.

22. Find the total number of different combinations of n things taken 1, 2, 3, ... n together, of which there are p of one sort, q of another, r of another, &c. Ans. $(p + 1)(q + 1)(r + 1)$. &c. $- 1$.

BINOMIAL THEOREM, ETC.

1. Expand $(a + b)^8$, $(a - b)^7$, $(2x - 3y)^5$, and $(5 - 4x)^4$.

(1) Ans. $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$.

(2) Ans. $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$.

(3) Ans. $32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$.

(4) Ans. $625 - 2000x + 2400x^2 - 1280x^3 + 256x^4$.

2. Required the coefficient of x^5 in the expansion of $(x+a)^8$. Ans. 56.

3. Expand $(\sqrt{a} \pm \sqrt{b})^4$. Ans. $a^2 + 6ab + b^2 \pm 4(a+b)\sqrt{ab}$.

4. Expand $\left(1 - \frac{x}{2}\right)^{-2}$. Ans. $1 + x + \frac{3}{4}x^2 + \frac{1}{2}x^3 + \frac{5}{16}x^4 + \ldots\ldots$

5. Find the 5th term of the expansion of $(a^2 - b^2)^{12}$. Ans. $495a^{16}b^8$.

6. Find the 7th term of the expansion of $(a^3 + 3ab)^9$.
Ans. $61236a^{15}b^6$.

7. Find the 5th term of the expansion of $(3a^2 - 7x^3)^8$.
Ans. $13613670a^8x^{12}$.

8. Find the 6th term of the expansion of $(ax + by)^{10}$.
Ans. $252a^5b^5x^5y^5$.

9. Find the middle term of the expansion of $(a^m + x^m)^{12}$.
Ans. $924a^{6m}x^{6m}$.

10. Find the middle term of $(a^{\frac{1}{8}} + b^{\frac{1}{8}})^8$. Ans. $70a^{\frac{1}{2}}b^{\frac{1}{2}}$.

11. Find the two middle terms of $(a + x)^{13}$. Ans. $1716a^7x^6$, $1716a^6x^7$.

12. Expand $\frac{1}{\sqrt[3]{ax - x^2}}$. Ans. $\frac{1}{(ax)^{\frac{1}{3}}} + \frac{1}{3}\cdot\frac{x^{\frac{2}{3}}}{a^{\frac{4}{3}}} + \frac{2}{9}\cdot\frac{x^{\frac{5}{3}}}{a^{\frac{7}{3}}} + \frac{14}{81}\cdot\frac{x^{\frac{8}{3}}}{a^{\frac{10}{3}}} + \ldots\ldots$

13. Expand $\frac{1}{(\sqrt[3]{a} - \sqrt[3]{x})^6}$. Ans. $\frac{1}{a^2} + \frac{6x^{\frac{1}{3}}}{a^{\frac{7}{3}}} + \frac{21x^{\frac{2}{3}}}{a^{\frac{8}{3}}} + \frac{56x}{a^3} + \ldots$

14. Find the middle term of the expansion of $(1 + x)^{2n}$.
Ans. $\frac{1.3.5 \ldots (2n-1)}{1.2.3 \ldots \quad n} . 2^n x^n$.

15. Find the r^{th} term of the expansion of $(3a - 2x)^{\frac{1}{5}}$.
Ans. $-\frac{1.4.9 \ldots (5r-11)}{1.2.3 \ldots \ (r-1)}\left(\frac{2x}{5}\right)^{r-1} . (3a)^{\frac{6}{5}-r}$.

16. Expand $\frac{1}{\sqrt[5]{a^5 - x^5}}$ to 5 terms; and find the $\overline{5+r}|^{\text{th}}$ term.

(1) Ans. $\frac{1}{a} + \frac{1}{5}\cdot\frac{x^5}{a^6} + \frac{3}{25}\cdot\frac{x^{10}}{a^{11}} + \frac{11}{125}\cdot\frac{x^{15}}{a^{16}} + \frac{44}{625}\cdot\frac{x^{20}}{a^{21}} + \ldots\ldots$

(2) Ans. $\frac{1.6.11.16 \ldots\ldots (16+5r)}{1.2.\ 3.\ 4 \ldots\ldots (4+r)} \cdot \frac{1}{5^{4+r}} \cdot \frac{x^{20+5r}}{a^{21+5r}}$.

17. Find the term which involves $a^{13}b^7$ in the expansion of $(a^3 + 3ab)^9$.
Ans. $78732a^{13}b^7$.

18. Expand $(a + 2b - c)^3$ by means of the Binomial Theorem.
Ans. $a^3 + 6a^2b + 12ab^2 + 8b^3 - 3a^2c - 12abc - 12b^2c + 3ac^2 + 6bc^2 - c^3$.

19. Find the term which involves a^4b^2 in $(7a^2 - 3ab + 4b^2)^3$.
Ans. $777a^4b^2$.

20. Find which is the greatest term of the expansion of $\left(1 + \frac{5}{6}\right)^{\frac{3}{2}}$.
Ans. The 2nd.

21. Find which is the greatest coefficient in $(1 + x)^{\frac{11}{\sqrt{2}}}$. Ans. The 5th.

22. At what term does the series for $\left(1 + \frac{9}{10}\right)^{\frac{3}{2}}$ begin to converge?
Ans. The 23rd.

23. Find an approximation to the cube root of 31 by the Binomial Theorem.
Ans. 3·14138.

24. Find an approximate value of $\sqrt[7]{108}$. Ans. 1·95204.

25. Find the sum of $1 + \frac{\alpha}{2} + \frac{\beta}{3} + \frac{\gamma}{4} + \&c.$ α, β, γ, &c. being the coefficients taken in order of the expansion of $(a + b)^n$.
Ans. $\frac{2^{n+1} - 1}{n + 1}$.

26. Prove that the coefficient of the $\overline{r + 1}^{\text{th}}$ term of $(1 + x)^{n+1}$ is equal to the sum of the coefficients of the r^{th} and $\overline{r + 1}^{\text{th}}$ terms of $(1 + x)^n$.

27. If r be the greatest whole number contained in $\frac{p}{q}$, $(a + x)^{\frac{p}{q}}$ has the first $\overline{r + 2}$ terms of its expansion positive, and the $\overline{r + m}^{\text{th}}$ of the same sign as $(-1)^m$; but $(a - x)^{\frac{p}{q}}$ has the first $\overline{r + 1}$ terms alternately positive and negative, and all the rest of the same sign as $(-1)^{r+1}$.

28. Given that the coefficient of the $\overline{p + 1}^{\text{th}}$ term of the expansion of $(1 + x)^n$ is equal to that of the $\overline{p + 3}^{\text{th}}$ term, shew that $p = \frac{n}{2} - 1$.

29. What is the meaning of $(1 + x)^{\sqrt{2}}$? Has the Binomial Theorem been proved in any sense to extend to such a quantity?

30. If $x > a$, prove that the sum of all the terms of the expansion of $(x + a)^n$, after the first two, is less than $(2^n - n - 1)\, ax^{n-1}$.

31. If $\frac{p(p-1)(p-2) \ldots\ldots (p-r+1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots r}$ be represented by p_r, prove that $(p + q)_r = p_r + p_{r-1}q_1 + p_{r-2}q_2 + \ldots\ldots + p_1q_{r-1} + q_r$.

32. If $\frac{n(n+1)(n+2) \ldots\ldots (n+r-1)}{1 \,.\, 2 \,.\, 3 \ldots\ldots r}$ be represented by ${}_nP_r$, shew that

$${}_nP_r + {}_{n+1}P_{r-1} = {}_{n+1}P_r.$$

33. Prove that

$$\frac{1.3.5\ldots(2r-1)}{\lfloor\!\underline{r}} + \frac{1.3.5\ldots(2r-3)}{\lfloor\!\underline{r-1}} \cdot \frac{3}{\lfloor\!\underline{1}} + \frac{1.3.5\ldots(2r-5)}{\lfloor\!\underline{r-2}} \cdot \frac{3.5}{\lfloor\!\underline{2}} + \ldots$$

is equal to $2^r(1+r)$.

34. Shew that if t_r denote the middle term of $(1+x)^{2r}$, then will

$$t_0 + t_1 + t_2 + \ldots\ldots = (1-4x)^{-\frac{1}{2}}.$$

35. If x be very small compared with 1, prove that

$$\frac{\sqrt{1+x} + \sqrt[3]{(1-x)^2}}{1 + x + \sqrt{1+x}} = 1 - \frac{5}{6}x \text{ nearly.}$$

36. If $c = a - b$, and is very small compared with a and b, shew that

$$\frac{a^2 b^2}{(a^2 - a^2x^2 + b^2x^2)^{\frac{3}{2}}} = a - 2c + 3cx^2 \text{ nearly.}$$

37. If $a > b$, prove that $a^n - b^n > nb^{n-1}(a-b)$ and $< na^{n-1}.(a-b)$.

38. If x be nearly equal to 1, shew that $\dfrac{mx^m - nx^n}{m-n} = x^{m+n}$ nearly.

39. If $u = \dfrac{a^p}{z^m} + \dfrac{b^q - a^p}{z^n}$, and z is very nearly equal to 1, shew that $u = b^q . z^{(n-m)\frac{a^p}{b^q} - n}$ very nearly.

40. If $\dfrac{x}{a} = 1 + h$, h being very small, find the value of

$$\frac{\left(2ax - x^2 + \dfrac{b^2x^2}{a^2}\right)^{\frac{1}{2}}}{(a^2+b^2)^{\frac{1}{2}}}.$$

Ans. $1 + \dfrac{b^2 h}{a^2 + b^2}$ nearly.

41. If N represent the n^{th} term of the expansion of a^x, find n when the series begins to converge at that term; and shew that the sum of all the succeeding terms is less than $\dfrac{Nn}{1 - x . \log a}$.

42. Required the term involving x^4 in the expansion of $(1 + x + x^2)^5$.

Ans. $45x^4$.

43. Required the term involving x^5 in the expansion of $(1 + 2x + 3x^2 + 4x^3 + \ldots)^6$.

Ans. $4368x^5$.

44. Required the term involving $a^2b^2c^2$ in the expansion of $(a+b+c)^6$.

Ans. $90a^2b^2c^2$.

45. Required the term involving $b^2c^3e^4f$ in the expansion of $(a + 2b + 3c + 4d + 5e + 6f)^{10}$.

Ans. $5103000000b^2c^3e^4f$.

46. Find the coefficient of x^4 in $(2+\sqrt{x}-x)^5$. Ans. 0.

47. Find the terms which involve $\frac{yz^9}{x^3}$, and y^5z^2 in $\left(\frac{x^2}{y}-\frac{y^2}{z}+\frac{z^3}{x}\right)^6$.
(1) Ans. $-\frac{6yz^9}{x^3}$, (2) Ans. $-60y^5z^2$.

48. Find the coefficient of $\frac{1}{x^8}$ in the expansion of $\left(a+\frac{b}{x}+\frac{c}{x^2}+\frac{d}{x^3}\right)^6$.
Ans. $6b^5d+15b^4c^2+120ab^3cd+60ab^2c^3+90a^2b^2d^2+15a^2c^4+60a^3cd^2+180a^2bc^2d$.

49. Find the coeff^t. of x^7 in the expansion of $(a+bx+cx^2+$ &c. *in inf.*$)^4$.
Ans. $12b^2cd+12a^2bg+12ac^2d+12a^2cf+12a^2cd+4a^3h$.

50. Find the coefficient of x^3 in $(a+bx+cx^2)^{\frac{3}{2}}$. Ans. $\frac{3}{4}\cdot\frac{bc}{a^{\frac{1}{2}}}-\frac{b^3}{16a^{\frac{3}{2}}}$.

51. Find the coefficient of x^n in the expansion of
$(1+x+2x^2+3x^3+\ldots)^2$. Ans. $\frac{1}{6}(n^3+11n)$.

52. Find the coefficient of x^r in the expansion of $(1+2x+3x^2+\ldots)^2$.
Ans. $\frac{4.5.6\ldots(r+3)}{1.2.3\ldots\ r}$.

53. Find the number of terms in the expansion of
$(a_0+a_1x+a_2x^2+a_3x^3)^4$. Ans. 35.

54. Find the number of terms in the expansion of $(a+b+c)^7$.
Ans. 36.

Write down the n^{th} term of each of the following series which proceed according to the law indicated by the terms given:—

1. $1+\frac{x}{1}+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots\ldots\ldots$ Ans. $\frac{x^{n-1}}{\underline{|n-1}}$.

2. $1+3+6+10+15+21+28+\ldots$ Ans. $\frac{1}{2}n(n+1)$.

3. $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\ldots$ Ans. $\frac{1}{\frac{1}{2}(n.\overline{n+1})}$.

4. $1.2.4+2.3.5+3.4.6+\ldots$ Ans. $n(n+1)(n+3)$.

5. $\frac{3\times5}{1^2\times3^2}+\frac{5\times7}{2^2\times4^2}+\frac{7\times9}{3^2\times5^2}+\ldots$ Ans. $\frac{(2n+1)(2n+3)}{n^2(n+2)^2}$.

6. $1+\frac{1.6.11}{1.2.3}+\frac{1.6.11.16.21}{1.2.3.4.5}+\ldots\ldots$ Ans. $\frac{1.6.11.16\ldots(10n-9)}{1.2.3.4\ \ldots\ (2n-1)}$.

7. $\frac{3}{5}+\frac{6}{8}+\frac{9}{11}+\frac{12}{14}+\ldots\ldots$ Ans. $\frac{3n}{3n+2}$.

8. $\frac{1}{6.7.8}+\frac{1}{8.9.10}+\frac{1}{10.11.12}+\ldots$ Ans. $\frac{1}{(2n+4)(2n+5)(2n+6)}$.

9. $\frac{19}{1.2.3}\times\frac{1}{4}+\frac{28}{2.3.4}\times\frac{1}{8}+\frac{39}{3.4.5}\times\frac{1}{16}+\ldots\ldots$

Ans. $\frac{(n+3)^2+3}{n(n+1)(n+2)}\times\frac{1}{2^{n+1}}$.

10. $1^3=1$, $2^3=3+5$, $3^3=7+9+11$, &c. Write down n^3 after the same law, and verify the result.

11. Find the first three terms of the series whose $\overline{n+1}^{\text{th}}$ term is $\frac{(n+7)(n-5)}{(n+1)(n^2+1)}.2^n$. Ans. $-35, -16, -7\frac{1}{5}$.

12. Find the $\overline{n+1}^{\text{th}}$ term of the series whose r^{th} term is $\frac{n(n-1)\ldots(n-r+2)}{1.2\ldots\ldots\ldots(r-1)}.\left(\frac{x}{1+2x}\right)^{r-1}$. Ans. $\left(\frac{x}{1+2x}\right)^n$.

13. Find the first three terms of the series whose r^{th} term is

$(-1)^{r+}.\frac{2^{\frac{3r-1}{3}}}{3^{2(r-1)}}.\frac{1.4.7\ldots.(3r-2)}{1.2.3\ldots\ldots r}$. Ans. $2^{\frac{2}{3}}$, $\frac{-2^{\frac{8}{3}}}{9}$, $\frac{7}{243}.2^{\frac{11}{3}}$.

EVOLUTION OF SURDS.

EXTRACT the square root of each of the following surds:—

1. $1\frac{7}{9}-2\frac{2}{3}\times\sqrt{\frac{1}{3}}$. Ans. $\frac{2}{3}(\sqrt{3}-1)$.

2. $\sqrt{27}+2\sqrt{6}$. Ans. $\sqrt[4]{12}+\sqrt[4]{3}$.

3. $\sqrt{32}-\sqrt{24}$. Ans. $\sqrt[4]{18}-\sqrt[4]{2}$.

4. $3\sqrt{5}+\sqrt{40}$. Ans. $\sqrt[4]{20}+\sqrt[4]{5}$.

5. $\left(a-\frac{b}{2}\right)^2+2\sqrt{a^3b-2a^2b^2+\frac{ab^3}{4}}$. Ans. $\sqrt{\left(a-\frac{b}{2}\right)^2-ab}+\sqrt{ab}$.

6. $\frac{3a}{x}+\sqrt{\frac{12a^3b^2}{c^2x}-\frac{4a^4b^4}{c^4}}$. Ans. $\frac{ab}{c}+\sqrt{\frac{3a}{x}-\frac{a^2b^2}{c^2}}$.

7. $ab+4c^2-d^2+2\sqrt{4abc^2-abd^2}$. Ans. $\sqrt{ab}+\sqrt{4c^2-d^2}$

8. Prove that $\sqrt{bc + 2b\sqrt{bc - b^2}} + \sqrt{bc - 2b\sqrt{bc - b^2}}$ is equal to $\pm 2b$.

9. Extract the cube root of $7 - 5\sqrt{2}$. Ans. $1 - \sqrt{2}$.

10. ... ... $45 \pm 29\sqrt{2}$. Ans. $3 \pm \sqrt{2}$.

11. ... ... $148 + 46\sqrt{11}$. Ans. $\sqrt[3]{2}\,\{2 + \sqrt{11}\}$.

12. Extract the 4th root of $14 + 8\sqrt{3}$. Ans. $\dfrac{\sqrt{3}+1}{\sqrt[4]{2}}$.

13. Extract the 5th root of $41 + 29\sqrt{2}$. Ans. $1 + \sqrt{2}$.

14. ... ... $228 + 132\sqrt{3}$. Ans. $(1 + \sqrt{3})\sqrt[5]{3}$.

15. Extract the 6th root of $2889 - 1292\sqrt{5}$. Ans. $2 - \sqrt{5}$.

16. Extract the 7th root of $239 + 169\sqrt{2}$. Ans. $1 + \sqrt{2}$.

17. Extract the square root of $9 + 2\sqrt{3} + 2\sqrt{5} + 2\sqrt{15}$.
Ans. $1 + \sqrt{3} + \sqrt{5}$.

18. Extract the square root of $6 + 2\sqrt{2} + 2\sqrt{3} + 2\sqrt{6}$.
Ans. $1 + \sqrt{2} + \sqrt{3}$.

19. Prove that $\sqrt{2 + \sqrt{8}}$ may be expressed by $\sqrt{1 + \sqrt{-1}} + \sqrt{1 - \sqrt{-1}}$. Does it follow that $\sqrt{2 + \sqrt{8}}$ is "impossible"?

20. Prove that $\sqrt{a \pm \sqrt{b}}$ may be reduced to the form $\sqrt[4]{\alpha} \pm \sqrt[4]{\beta}$, when $1 - \dfrac{a^2}{b}$ is a complete square.

21. Prove that $(20 + \sqrt{392})^{\frac{1}{3}} + (20 - \sqrt{392})^{\frac{1}{3}}$ is equal to 4.

22. If $x = \sqrt[3]{-\dfrac{r}{2} + \sqrt{\dfrac{r^2}{4} - \dfrac{q^3}{27}}} + \sqrt[3]{-\dfrac{r}{2} - \sqrt{\dfrac{r^2}{4} - \dfrac{q^3}{27}}}$, and $r = 0$, shew that the values of x are 0 and $\pm\sqrt{q}$.

23. If $\left(x - \dfrac{y}{p}\right).\{b - y - (a - x)p\} = c^2$, and $p = \sqrt{\dfrac{y^2 - by}{x^2 - ax}}$, prove that $\sqrt{(b - y)x} - \sqrt{(a - x)y} = c$.

24. Given that $2\{x^2 + y^2 - x - y\} + 1 = 0$; required x and y.
Ans. $x = \dfrac{1}{2}$, $y = \dfrac{1}{2}$.

25. Given $(x + y\sqrt{-1})^2 = a + b\sqrt{-1}$; find x and y.
Ans. $x = \pm\sqrt{2a \pm \sqrt{4a^2 + b^2}}$, $y = \pm\sqrt{a \pm \sqrt{4a^2 + b^2}}$.

26. Given $\dfrac{3x + 2y\sqrt{-1}}{5\sqrt{-1} - 2} = \dfrac{15}{8x + 3y\sqrt{-1}}$, find x and y. Ans. $\begin{cases} x = 1, \\ y = 3. \end{cases}$

INDETERMINATE COEFFICIENTS.

By the method of Indeterminate Coefficients shew that,

1. $\frac{3+2x}{5+7x}=\frac{3}{5}-\frac{11}{5^2}x+\frac{7\times 11}{5^3}x^2-\frac{7^2\times 11}{5^4}x^3+\frac{7^3\times 11}{5^5}x^4-\ldots$

2. $\frac{1+x}{(1-x)^3}=1^2+2^2.x+3^2.x^2+4^2.x^3+5^2.x^4+\ldots$

3. $\frac{1}{x^2-(a+b)x+ab}=\frac{1}{(a-b)(x-a)}-\frac{1}{(a-b)(x-b)}.$

4. $\frac{3x-5}{x^2-6x+8}=\frac{7}{2}.\frac{1}{x-4}-\frac{1}{2}.\frac{1}{x-2}.$

5. $\frac{3x^2-7x+6}{(x-1)^3}=\frac{2}{(x-1)^3}-\frac{1}{(x-1)^2}+\frac{3}{x-1}.$

6. $\frac{1}{a^4-x^4}=\frac{1}{4a^3(a+x)}+\frac{1}{4a^3(a-x)}+\frac{1}{2a^2(a^2+x^2)}.$

7. $\frac{x-bx^3+dx^5}{1-ax^2+cx^4}=x+(a-b)x^3+(a^2-ab-c+d)x^5+\ldots$

8. $\frac{1}{x^6-1}=\frac{1}{6}\left\{\frac{1}{x-1}-\frac{1}{x+1}+\frac{x-2}{x^2-x+1}-\frac{x+2}{x^2+x+1}\right\}.$

9. $\frac{x^2+px+q}{(x-a)(x-b)(x-c)}=\frac{a^2+pa+q}{(a-b)(a-c)(x-a)}+\frac{b^2+pb+q}{(b-a)(b-c)(x-b)}$
$+\frac{c^2+pc+q}{(c-a)(c-b)(x-c)}.$

10. $\sqrt{1+x+x^2+x^3+\ldots\ldots}=1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\ldots\ldots$

11. Resolve $7x^2-6x-1$ into two factors of the first degree.
Ans. $(x-1)(7x+1)$.

12. Resolve $2x^2-21xy-11y^2-x+34y-3$ into factors of the first degree.
Ans. $(x-11y+1)(2x+y-3)$.

13. Given $y=ax+bx^2+cx^3+dx^4+\ldots$, find x in a series of powers of y.
Ans. $x=\frac{1}{a}y-\frac{b}{a^3}y^2+\frac{2b^2-ac}{a^5}y^3-\frac{5b^3-5abc+a^2d}{a^7}y^4+\ldots$

14. Given $y^3-axy-b^3=0$, find y in a series of powers of x.
Ans. $y=b+\frac{ax}{3b}-\frac{a^3x^3}{3^4b^5}+\frac{a^4x^4}{3^6b^7}-\ldots$

15. Given $x=n-\frac{1}{2}n^2+\frac{1}{3}n^3-\frac{1}{4}n^4+\ldots$, find n in a series of powers of x.
Ans. $n=x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\ldots..$

CONTINUED FRACTIONS.

1. Find a series of fractions converging to $\frac{251}{764}$; also to $\frac{39}{139}$.

(1) Ans. $\frac{1}{3}, \frac{22}{67}, \frac{23}{70}, \frac{114}{347}$, &c. (2) Ans. $\frac{1}{3}, \frac{1}{4}, \frac{2}{7}, \frac{7}{25}, \frac{16}{57}$.

2. Express $\frac{84}{227}$ in a continued fraction, and find the convergents.

Ans. The quotients are 2, 1, 2, 2, 1, 3, 2.

The convergents are $\frac{1}{2}, \frac{1}{3}, \frac{3}{8}, \frac{7}{19}, \frac{10}{27}, \frac{37}{100}, \frac{84}{227}$.

3. Find the convergents to the continued fraction whose quotients are 1, 4, 9, 2, 1, 1, 4.

Ans. $\frac{1}{1}, \frac{5}{4}, \frac{46}{37}, \frac{97}{78}, \frac{143}{115}, \frac{240}{193}, \frac{1103}{887}$.

4. Find a series of fractions converging to $\sqrt{2}$; also to $\sqrt{45}$.

(1) Ans. $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}$, &c. (2) Ans. $\frac{6}{1}, \frac{7}{1}, \frac{20}{3}, \frac{47}{7}, \frac{114}{17}$, &c.

5. Express $\frac{3+\sqrt{7}}{2}$ in a continued fraction.

Ans. The quotients are 2, 1, 4, 1, 1, 1, 4, &c.

6. Find the convergents to 0·2422638.....

Ans. $\frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{39}{161}, \frac{47}{194}$, &c.

7. From the last example deduce an explanation of the Julian and Gregorian corrections of the Calendar, having given the true length of the year to be 365·2422638......

8. Prove that $\frac{355}{113}$ differs from 3·14159 by a quantity less than 0·00001.

9. The lunar month, calculated on an average of 100 years, is 27·321661 days. Find a series of vulgar fractions approximately nearer and nearer to this decimal fraction.

Ans. $\frac{27}{1}, \frac{82}{3}, \frac{765}{28}, \frac{3907}{143}$, &c.

10. The sidereal revolution of Mercury is 87·969255 days; and that of Venus 224·700817 days. Represent these quantities approximately by less numbers.

Ans. For Mercury $\frac{87}{1}, \frac{88}{1}, \frac{2815}{32}$, &c.

For Venus $\frac{224}{1}, \frac{225}{1}, \frac{674}{3}, \frac{1573}{7}$, &c.

11. Find the least fraction with only two figures in each term, approximating to $\frac{1947}{3359}$. Ans. $\frac{11}{19}$.

12. Given $2^x = 6$. Required x in the form of a continued fraction; and find the convergents.

Ans. $x = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \dots}}}$ Convergents are $\frac{2}{1}, \frac{3}{1}, \frac{5}{2}, \frac{13}{5}, \frac{31}{12}$, &c.

13. If $\left(\frac{7}{12}\right)^x = \frac{3}{4}$, find x. Ans. 0·53.

14. If $3^x = 15$, find x. Ans. 2·465.

15. If $\left(\frac{1}{2}\right)^x = \frac{3}{5}$, find x. Ans. 0·737.

16. Approximate by continued fractions to the roots of the equations,

(1) $5x^2 - 3 = 0$, (2) $x^2 - 5x + 3 = 0$.

(1) Ans. $\frac{3}{5}, \frac{4}{5}, \frac{27}{35}, \frac{31}{40}, \frac{213}{275}, \frac{244}{315}$, &c.

(2) Ans. $\frac{4}{1}, \frac{13}{3}, \frac{43}{10}, \frac{142}{33}$, &c. and $\frac{1}{2}, \frac{3}{4}, \frac{2}{3}, \frac{7}{10}, \frac{23}{33}$, &c.

17. Shew that $\sqrt{5}$ is greater than $\frac{682}{305}$ and less than $\frac{2889}{1292}$, and that it differs from the latter fraction by a quantity less than $\frac{1}{2 \times 305 \times 1292}$.

INDETERMINATE EQUATIONS AND PROBLEMS.

1. $14x - 5y = 7$; find the least positive integral values of x and y. Ans. $x = 3$, $y = 7$.

2. $27x + 16y = 1600$; .... Ans. $x = 48$, $y = 19$.

3. $19x - 117y = 11$; .... Ans $x = 56$, $y = 9$.

4. $3x + 5y = 26$; find *all* the values of x and y in positive integers. Ans. $x = 7, 2$. $y = 1, 4$.

5. $11x + 13y = 190$; ... ... Ans. $x = 9$, $y = 7$.

6. $13x + 16y = 97$; ... ... Ans. $x = 5$, $y = 2$.

7. $11x+7y=108$; find *all* the values of x and y in positive integers.
Ans. $x=6,\ y=6$.

8. Shew that there is no solution in whole numbers for the equation $49x-35y=11$.

9. Given $x=4,\ y=9$, one solution of $2x+3y=35$. Find *all* the solutions in positive integers.

$$\text{Ans.}\ \begin{cases} x=1,\ 4,\ 7,\ 10,\ 13,\ 16. \\ y=11,\ 9,\ 7,\ 5,\ 3,\ 1. \end{cases}$$

10. Given $\left.\begin{aligned} x+2y+3z&=20 \\ \text{and}\ 4x+5y+6z&=47 \end{aligned}\right\}$; find x, y, z.

$$\text{Ans.}\ \begin{cases} x=1,\ 2,\ 3, \\ y=5,\ 3,\ 1, \\ z=3,\ 4,\ 5. \end{cases}$$

11. Given $\left.\begin{aligned} 6x+7y+4z&=122 \\ \text{and}\ 11x+8y-6z&=145 \end{aligned}\right\}$; find x, y, z.
Ans. $x=9,\ y=8,\ z=3$.

12. Find all the positive integral solutions of $20x-21y=38$, and $3y+4z=34$.
Ans. $x=4,\ y=2,\ z=7$.

13. Find all the positive integral solutions of $xy+x^2=2x+3y+29$.
Ans. $x=4,\ 5,\ y=21,\ 7$.

14. Find all the positive integral solutions of $7xy-5x=3y+39$.

$$\text{Ans.}\ \begin{cases} x=1,\ 3,\ 5,\ 21. \\ y=11,\ 3,\ 2,\ 1. \end{cases}$$

15. Find the number of solutions of $11x+15y=1031$ in positive integers.
Ans. 7.

16. Find the number of solutions of $3x+7y+17z=100$ in positive integers.
Ans. 10.

17. Find the number of solutions of $20x+15y+6z=171$ in positive integers.
Ans. 12.

18. Find two fractions having 7 and 9 for their denominators, and their sum $\frac{57}{63}$.
Ans. $\frac{4}{7},\ \frac{3}{9}$.

19. Find three fractions with denominators 3, 4, and 5, of which the sum is $\frac{133}{60}$.
Ans. $\frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}$.

20. Find the three fractions whose sum is $\frac{401}{315}$.
Ans. $\frac{2}{5},\ \frac{3}{7},\ \frac{4}{9}$.

21. Find a number which upon being divided by 39 gives a remainder 16, and by 56 a remainder 27.
Ans. 1147, or 3331, &c.

22. Find the least number which, upon being divided by 11, 19, and 29, gives the remainders 3, 5, and 10, respectively. Ans. 4128.

23. Find a number less than 400 which is a multiple of 7, and upon being divided by 2, 3, 4, 5, 6, always leaves 1 for a remainder. Ans. 301.

24. Shew that the solution of $ax + by = c$ in positive integers is always possible, if $c > ab - (a + b)$.

25. In how many different ways is it possible to pay £20 in half-guineas and half-crowns? Ans. 7.

26. A certain sum consists of x£. y shillings, and its half of y£. x shillings; find the sum. Ans. £13. 6*s*.

27. Find two numbers such that their sum shall be equal to the sum of their squares. Ans. $\frac{15}{13}$, $\frac{3}{13}$.

28. What value of x will make $ax^2 + bx + c^2$ a complete square?

Ex. $2x^2 + x + 8$. (1) Ans. $x = \dfrac{bn^2 - 2cmn}{m^2 - an^2}$. (2) Ans. 8, or -4.

29. What value of b will make $b^2 - 4ac$ a complete square?

Ans. $b = am + \dfrac{c}{m}$.

30. Find three square numbers which are in Arithmetic Progression.

Ans. $(m^2 - n^2 - 2mn)^2$, $(m^2 + n^2)^2$, $(m^2 - n^2 + 2mn)^2$.

SCALES OF NOTATION.

1. 17486 is in the denary scale, find the equivalent number in the senary scale. Ans. 212542.

2. 215855 is in the denary scale, find the same number in the duodenary scale. Ans. *t*4*tee*.

3. 3*t*4*e*2 is in the duodenary scale, find the same number in the denary scale. Ans. 80198.

4. Transform 1534 from the senary to the denary scale. Ans. 418.

5. Divide 14332216 by 6541 in the septenary scale. Ans. 1456.

6. Divide 95088918 by *tt*4 in the duodenary scale. Ans. *t*4*tee*.

7. Multiply 64 ft. 6 in. by 8 ft. 9¼ in. Ans. 565 ft. 8′. 7″. 6‴.

8. The difference between any number (in the denary scale) and that formed by reversing the order of the digits is divisible by 9. Prove it.

9. Extract the square root of 25400544 in the senary scale.

Ans. 4112.

10. Extract the square root of $32e75721$ in the duodenary scale; and then verify the result by squaring it.

11. The number 124 in the denary is expressed by 147 in another scale, required the radix of the latter. Ans. 9.

12. In what scale of notation will a number that is double of 145 be expressed by the same digits? Ans. Radix = 15.

13. Find the scale to which 24065 belongs, its equivalent in the denary scale being 6221. Ans. Radix = 7.

14. The area of a rectangle is 29ft. 0in. $4'$., and its length is 12ft. 8in.; find its breadth. Ans. 2ft. 3in. $6'$.

15. The area of a rectangle is 971ft. 10in., and breadth 24ft. 9in.; find its length. Ans. 39ft. 3in. $2'.3''$.

16. The area of a square is 17ft. 4in. $6'$., what is the length of the side? Ans. 4ft. 2in. $0'.2''.10'''$.

17. What is the cube of 6ft. 6in.? Ans. 188ft. 1090in.

18. Prove that any number of 4 digits in the denary scale is divisible by 7, if the first and last digits be the same, and the digit in the place of hundreds be double that in the place of tens.

19. Any number is divisible by 4, if the last two digits, taken in order to form a number, be divisible by 4.

20. Any number is divisible by 8, if the number, consisting of the last three digits in order, be divisible by 8.

21. There is a certain number consisting of two digits, which is equal to four times the sum of its digits; and if to the number 18 be added, the digits will be reversed. Ans. 24.

22. There is a certain number, a multiple of 10, which exceeds the sum of its digits by 99; find the number. Ans. 100.

23. Prove that the sum of all the numbers which are composed of the same digits is divisible by the sum of the digits.

24. Find the greatest and the least numbers of 4 digits in the senary scale, as expressed in the denary scale. Ans. 1295, 216.

25. The two digits of which a number consists are as 3 : 2, and when the digits are reversed the number is divisible by 3, and is to the former number as 23 : 32. Required the number. Ans. 96.

26. Any number consisting of an even number of digits, in a system whose radix is r, is divisible by $r+1$, if the digits equidistant from each end are the same.

27. If N, N', be any two numbers in the denary scale composed of the same digits differently arranged, prove that $N \sim N'$ is divisible by 9.

28. The square of any number which has less than 10 digits, (in the denary scale) each of which is 1, will, when reckoned from either end, form the same Arithmetic Progression whose common difference is 1, and greatest term the number of digits in the root

PROPERTIES OF NUMBERS.

1. PROVE that n^3 divided by 4 cannot leave 2 for a remainder, n being any of the natural numbers.

2. No number can be a square which has any one of the numbers 2, 3, 7, 8 for its last digit.

3. Prove the following properties of a square number:—

(1) A square number cannot terminate with an odd number of ciphers.

(2) If a square number terminate with 5, it must terminate with 25.

(3) If a square number terminate with an odd figure, the last figure but one will be even; and if it terminate with any even figure except 4, the last figure but one will be odd.

(4) No square number can terminate with two figures the same, except they be two ciphers, or two 4's.

4. Any number divided by 6 leaves the same remainder as its cube divided by 6.

5. If m be any odd square number greater than 1, prove that $(m+3) \times (m+7)$ is divisible by 32.

6. If each of the quantities a, b, n be a whole number, shew that $\{2a+(n-1)b\}.\frac{n}{2}$ is always a whole number.

7. Shew that every perfect cube number is of one of the forms $4n$, or $4n \pm 1$.

8. Shew that $x^5 - 5x^3 + 4x$ is divisible by 120, whatever positive whole number x may be.

9. The difference of the squares of any two odd numbers is divisible by 8; and the difference of the squares of any two prime numbers, of which the less exceeds 5, is divisible by 24.

10. If n be any whole number, one of the three n^2, n^2+1, n^2+2, is divisible by 5.

11. If n be an odd number not divisible by 7, either n^3+1, or n^3-1 is divisible by 14.

12. Shew that, when m is any even number greater than 2, $m^2(m^2-4)$ is divisible by 192; and, when m is any odd number greater than 3, $m(m^2-1)(m^2-9)$ is divisible by 1920.

13. If n be a prime number greater than 3, $\dfrac{n^2-1}{24}$ is an integer.

14. If a and b be prime numbers, the number of numbers prime to ab and less than ab is equal to $(a-1)(b-1)-1$.

15 If there be two binomials each of which is the sum of two squares, their product is the sum of two squares.

16. Neither the sum nor the difference of two irreducible fractions, whose denominators are different, can be an integer.

17. If n be any number, and a the difference between n and the next greater square number, and b the difference between n and the next less square number, shew that $n-ab$ is a square.

18. Prove that the products of two different primes cannot be a square.

19. Decompose 831600 into its prime factors; and find the multiplier which will make it a perfect cube.

(1) Ans. $11 \times 7 \times 5^2 \times 2^4 \times 3^3$. (2) Ans. 118580.

20. Find the number of divisors of 1000. Ans. 16.

21. Find the number of divisors of 30030. Ans. 64.

22. If N is a number of the form $a^m b^n$, where a and b are prime numbers, shew that $N.\dfrac{a-1}{a}.\dfrac{b-1}{b}$ is the number of integers not greater than N and prime to it.

How many numbers are there not greater than 100 and prime to it? Ans. 40.

How many less than 360 and prime to it? Ans. 96.

23. If one number (A) have exactly as many places of figures as another (B), and also have more than the first half of its figures identical with the corresponding figures in B, shew that the difference between $\sqrt[n]{A}$ and $\sqrt[n]{B}$ will be less than $\dfrac{1}{n}$, if n be any whole number not less than 2.

VANISHING FRACTIONS.

1. FIND the value of $\dfrac{\sqrt{x}-\sqrt{a}+\sqrt{x-a}}{\sqrt{x^2-a^2}}$ when $x=a$. Ans. $\dfrac{1}{\sqrt{2a}}$.

2. Find the value of $\dfrac{a^3-b^3}{a^2-b^2}$ when $a=b$. Ans. $\dfrac{3a}{2}$.

3. Find the value of $\dfrac{a^m - b^m}{a^n - b^n}$ when $a = b$. Ans. $\dfrac{m}{n} a^{m-n}$.

4. Find the value of $\dfrac{a - (a^n - x^n)^{\frac{1}{n}}}{x^n}$ when $x = 0$. Ans. $\dfrac{1}{n a^{n-1}}$.

5. Find the value of $\dfrac{\sqrt{2a^3x - x^4} - \sqrt{ax^3}}{a - \sqrt{ax}}$ when $x = a$. Ans. $5a$.

6. Find the value of $\dfrac{x^3 + 5ax^2 - 4a^2x - 2a^3}{x^2 - a^2}$ when $x = a$. Ans. $\dfrac{9a}{2}$.

7. Find the value of $\dfrac{\sqrt{a^2 + ax + x^2} - \sqrt{a^2 - ax + x^2}}{\sqrt{a + x} - \sqrt{a - x}}$ when $x = 0$.

Ans. $\sqrt{a}$.

8. Find the value of $\dfrac{\sqrt{x^2 + a} - \sqrt{a^2 + x} + \sqrt{a^2 - x^2}}{\sqrt{a^3 - x^3}}$ when $x = a$.

Ans. $\sqrt{\dfrac{2}{3a}}$.

CONVERGING AND DIVERGING SERIES.

1. **Prove** that $1 + \dfrac{x}{1} + \dfrac{x^2}{1.2} + \dfrac{x^3}{1.2.3} + \ldots\ldots$ will begin to converge at some point for any value of x, however great.

2. Shew that the series for $(1 + x)^n$, given by the Binomial Theorem, will always be "convergent" when $x < 1$; and determine after how many terms the convergency will begin.

3. Shew that both the following series are convergent,

$$(1)\ 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \ldots\ldots \qquad (2)\ x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \ldots\ldots$$

and if $x = 1000$, at which term will the former series begin to converge?

Ans. $\dfrac{x^{264}}{\underline{|264}}$.

4. In the expression $ax^n + bx^{n-1} + cx^{n-2} + \ldots\ldots$ if a be any fixed quantity however small, and b, c, d, &c. any fixed quantities however great, shew that x may be taken so great that ax^n shall contain $bx^{n-1} + cx^{n-2} + \ldots\ldots$ as many times as we please.

5. How small must x be taken, so that the third term of the infinite series $1 + 3x + 5x^2 + 7x^3 + \ldots\ldots$ may contain the sum of all that follows 500 times?

Ans. $x < \dfrac{1}{702}$.

LOGARITHMS.

Apply the Logarithmic Tables to find,

1. 2^{64}. Ans. 18446750000000000000.

2. $\dfrac{239 \times 827 \times 543}{76 \times 17}$. Ans. 83069·32.

3. $\sqrt[6]{235{\cdot}78}$. Ans. 2·485522…

4. $\left(\dfrac{9}{8}\right)^{21}$. Ans. 11·86322…

5. $\sqrt[5]{\dfrac{7}{3}\sqrt[4]{6}}$. Ans. 1·295695…

6. $\sqrt[4]{\dfrac{132 \times (7{\cdot}356)^9}{\sqrt{(3{\cdot}25)^5}}}$. Ans. 144·5972…

7. $\dfrac{(1{\cdot}05)^7 - 1}{(1{\cdot}05)^7 \times 0{\cdot}05}$. Ans. 5·79…

8. $\sqrt[5]{\left(\dfrac{23}{417}\right)^3}$. Ans. 0·17577…

9. £15. 7*s.* 3½*d.* × $(1{\cdot}03)^{50}$. Ans. £67. 7*s.* 1*d.*

10. Given $20^x = 100$, find x. Ans. $x = 1{\cdot}537244$.

11. Given $c^{mx} = a\,.\,b^{nx-1}$, find x. Ans. $x = \dfrac{\log a - \log b}{m \log c - n \log b}$.

12. Given $a^{4x} + a^{2x} = a^{6x}$, find x. Ans. $x = \dfrac{\log \frac{1}{2}(1 + \sqrt{5})}{2 \log a}$.

13. Given $a^x + \dfrac{1}{a^x} = b$, find x. Ans. $x = \dfrac{\log \frac{1}{2}(b \pm \sqrt{b^2 - 4})}{\log a}$.

14. Given $(\sqrt[r]{a^c})^x = b^{cx-4a}$, find x. Ans. $x = \dfrac{4ar\,.\log b}{c(r \log b - \log a)}$.

15. Given $x^y = y^x$, and $x^p = y^q$, find x and y.

Ans. $\log x = \dfrac{q}{q-p}\{\log p - \log q\}$, $\log y = \dfrac{p}{q-p}\{\log p - \log q\}$.

16. Given $3^{2x}\,.\,5^{3x-4} = 7^{x-1}\,.\,11^{2-x}$, find x. Ans. $x = 1{\cdot}242076\ldots$

17. Given $(a^4 - 2a^2b^2 + b^4)^{x-1} = \dfrac{(a-b)^{2x}}{(a+b)^2}$, find x. Ans. $x = \dfrac{\log \overline{a-b}}{\log \overline{a+b}}$.

18. Given $5^{x+1} - 5^{x-3} + 14\frac{1}{9} = 4739 - 5^{x-4} + \frac{2}{7} - 5^{x-2}$, find x. Ans. 1·106.

19. Given $\left.\begin{array}{r} 5 \times 3^x - 3 \times 2^y = 30000, \\ \text{and } 3 \times 3^x + 6 \times 2^y = 20000, \end{array}\right\}$ find x and y. Ans. $\left\{\begin{array}{l} x = 8{\cdot}55. \\ y = 8{\cdot}533. \end{array}\right.$

20. Given $3^{x^2-4x+5} = 1200$, find x. Ans. $x = 4{\cdot}33$, or $-0{\cdot}33$.

21. Given $a^1 . a^3 . a^5 . a^7 .$ &c. $= p$, find the number of factors a^1, a^3, a^5, &c.

Ans. $\sqrt{\dfrac{\log p}{\log a}}$.

INTEREST AND ANNUITIES.

1. WHAT principal put out at simple interest for 3 years, at the rate of 5 per cent. per annum, will amount to £828? Ans. £720.

2. A person borrows £450 at 5 per cent. simple interest, and returns for it £517. 10*s.*; for what time was it lent? Ans. 3 years.

3. A person returns £610 for the loan of £600 for one month, what is the rate of interest allowed. Ans. 20 per cent.

4. What will a capital of £12000 amount to in 10 years, at 6 per cent. per annum compound interest, the interest being paid half-yearly? Ans. £21673. 6*s.* 8*d.*

5. Prove that the amount of £1 in n years at compound interest is given within less than a farthing by the first four terms of the expansion of $(1 + r)^n$£, the rate of interest being not greater than 4 per cent., and n not greater than 10.

6. Find in what time a sum of money will double itself, put out at 5 per cent. per annum, reckoning compound interest. Ans. 14·2 years.

7. Find in what time at compound interest, reckoning 5 per cent. per annum, £100 will amount to £1000. Ans. 47·14 years.

8. What is the amount of one farthing, for 500 years, at 3 per cent. per annum, compound interest? Ans. £2731. 2*s.* 5¼*d.*

9. Shew that the common rule for determining the equated time of payment of several sums due at different times is in favour of the payer.

10. Required the discount on £160 for a quarter of a year, reckoning interest at the rate of 5 per cent. per annum. Ans. £1. 19*s.* 6*d.*

11. What will be the amount of £1212 per annum left unpaid for 76 years, reckoning 4 per cent. per annum, compound interest? Ans. £566702. 17*s.* 4*d.*

12. What will an annuity of £250 amount to in 7 years, paid half-yearly, allowing 6 per cent. per annum simple interest? Ans. 2091. 5*s*.

13. What annuity improved at the rate of 8 per cent. per annum, compound interest, will at the end of 10 years amount to £3000?

Ans. £207. 1*s*. 9*d*.

14. What is the present value of an annuity of £20 to continue for 40 years, reckoning interest at the rate of 6 per cent. per annum?

Ans. £300. 18*s*. 6*d*.

15. An annuity of £20 for 21 years is sold for £220; required the rate of interest allowed to the purchaser. Ans. £6. 16*s*. 5*d*. per cent.

16. If a lease of $55\frac{1}{4}$ years be purchased for £100, what rent ought to be received, that the purchaser may make $5\frac{1}{2}$ per cent. per annum for his money? Ans. £5. 16*s*.

17. An annuity A is to commence at the end of p years, and to continue q years; find the equivalent annuity to commence immediately and to continue q years.

Ans. $\dfrac{A}{(1+r)^p}$.

18. The discount on a promissory note of £100 amounted to £7. 10*s*. and the interest made by the banker was £5·405405... per cent.; find the interval at the end of which the note was payable. Ans. $1\frac{1}{2}$ years.

19. A person puts out P£ at interest, and adds to his capital at the end of every year $\frac{1}{m}$th part of the interest for that year; find the amount at the end of n years.

Ans. $P.\left\{\dfrac{mr+m+r}{m}\right\}^n$.

20. The lease of an estate is granted for 7 years at a pepper-corn rent, with the condition that the tenant at the expiration of the lease may renew the same on paying a fine of £100. What is the value of the landlord's interest in the estate immediately after any such renewal, allowing compound interest at the rate of 5 per cent. per annum?

Ans. £245. 12*s*. 10*d*.

21. A person spends in the first year m times the interest of his property; in the second $2m$ times that of the remainder; in the third $3m$ times that of what remained at the end of the second; and so on. At the end of $2p$ years he has nothing left. Shew that in the p^{th} year he spends as much as he has left at the end of that year.

22. The reversion of an estate in fee simple producing £60 a year is made over for the discharge of a debt of £577. 4*s*. 5*d*. How soon ought the creditor to take possession, if he be allowed 5 per cent. per annum interest for his debt? Ans. 15 years.

23. A person puts his whole fortune P£ out at interest, at the rate of £ per £1 per annum, and requires for his annual expences p£ more than he whole interest of P£. In how many years, at this rate, will he have pent the whole? and if, when he has spent half his capital, he diminishes his expenditure one half, how much longer on this than on the ormer supposition will he continue solvent?

(1) Ans. $\dfrac{\log \overline{Pr + p} - \log p}{\log \overline{1 + r}}$. (2) Ans. $\dfrac{\log\left(\dfrac{Pr}{2} + p\right) - \log p}{\log 1 + r}$.

24. Find the present value of an annuity of £20 a year, to commence n 10 years, and then to continue 11 years, reckoning 4 per cent. per nnum compound interest. Ans. £118. 7*s.* 3½*d.*

25. What sum ought to be paid for the Reversion of an annuity of £50 for 7 years after the next 14, that the purchaser may make 6 per cent. per annum of his money? Ans. £123. 9*s.* 1*d.*

26. If a person purchase the Reversion of an estate after 20 years for £500, what rent ought it to produce that he may make 6 per cent. per nnum of his money? Ans. £96. 4*s.* 3*d.*

27. A debt of a£ accumulating at compound interest is discharged in years by annual payments of $\dfrac{a}{m}$£; prove that $n = -\dfrac{\log(1 - mr)}{\log(1 + r)}$.

28. If M_s, M_c represent the sums to which an annuity would amount n n years at simple and compound interest respectively, prove that

$$\frac{M_s}{M_c} = \frac{nr}{2} \cdot \frac{(n - 1)r + 2}{(1 + r)^n - 1}.$$

29. If two joint-proprietors have an equal interest in a freehold estate worth p£ per annum, but one of them purchase the whole to himself by allowing the other an equivalent annuity of q£ for n years, shew that

$$\frac{p}{q} = 2\left(1 - \frac{1}{(1 + r)^n}\right).$$

30. If P represent the population of any place at a certain time, and every year the number of deaths is $\dfrac{1}{p}$th, and the number of births $\dfrac{1}{q}$th, of the whole population at the beginning of that year; required the amount of the population at the end of n years from that time.

Ans. $P\left\{1 + \dfrac{p - q}{pq}\right\}^n$.

31. In the last problem, if $p = 60$, and $q = 45$, shew that the population will be doubled in 125 years, nearly.

32. What must be the annual increase in the population of any country, that it may double itself in a century?

Ans. Between $\dfrac{1}{143}$rd and $\dfrac{1}{144}$th.

CHANCES, AND LIFE ANNUITIES.

1. **What** is the chance of drawing the four aces from a pack of cards in 4 successive trials?

Ans. $\frac{1}{270725}$.

2. There are 4 white balls and 3 black placed at random in a line, find the chance of the extreme balls being both black.

Ans. $\frac{1}{7}$.

3. Of two bags one contains 9 balls, and the other 6, and in each bag the balls are marked *a, b, c, d*, &c. If one be drawn from each bag, what is the chance that the two will have the same letter-mark?

Ans. $\frac{1}{9}$.

4. *A* plays one game with *B*, and another with *C*; the odds that he does not win both games are 4 to 1; the odds that he beats *B* are *3 to 2*. What are the odds in his game with *C*? Ans. Odds against him 2 to 1.

5. From a common pack of cards 12 are dealt to as many persons, one to each, the cards collected, shuffled, and the same repeated. What is the chance that a given person will on both occasions have the same card dealt to him? What is the chance that the two cards dealt to him will have the sum of their numbers equal to 3?

(1) Ans. $\frac{1}{52}$. (2) Ans. $\frac{2}{169}$.

6. Two dice are placed together at random so as to form a parallelopiped: determine the chance that two or more adjacent faces will have the same marks.

Ans. $\frac{7}{24}$.

7. From a bag containing 2 guineas, 3 sovereigns, and 5 shillings, a person is allowed to draw 3 of them indiscriminately: what is the value of his expectation? Ans. $32\frac{1}{10}$ shillings.

8. A shilling is thrown upon a chess-board, a square of which will just include 4 shillings, find the chance of its falling clear of a division.

Ans. $\frac{1}{4}$.

9. Three men, *A, B, C*, in succession throw a die, on condition that he who first throws an ace shall receive £1; what are the values of their several expectations?

Ans. *A*'s $= 7s.\ 10\frac{86}{91}d.$ *B*'s $= 6s.\ 7\frac{11}{91}d.$ *C*'s $= 5s.\ 5\frac{85}{91}d.$

10. There are 5 persons, out of which 4 are going to play at whist. They all cut, and the lowest sits out. What is the chance that two specified individuals will be partners?

Ans. $\frac{1}{5}$.

11. There is a lottery containing black and white balls, from each drawing of which it is as likely a black ball shall arise as a white one, what is the chance of drawing 11 balls all white?

Ans. $\frac{1}{2048}$.

12. There is a lottery of 10 green, 12 white, and 14 red balls. Let 2 have been drawn, what is the probability that they will be green and white?

Ans. 17 to 4 against it.

13. A die is thrown time after time: in how many times have we an even chance of throwing an ace?

Ans. Not quite an even chance in 3, but more than even in 4, times.

14. Two witnesses, on each of whom it is 3 to 1 that he speaks truth, agree in affirming that a certain event did happen, which of itself is equally likely to have happened or not. What is the chance that the event did happen?

Ans. 9 to 1.

15. *A* speaks truth 3 times out of 4, *B* 4 times out of 5, *C* 6 times out of 7; what is the probability of the truth of what *A* and *B* agree in asserting, but which *C* denies?

Ans. 2 to 1.

16. Thirteen persons are required to take their places at a round table by lot; shew that it is 5 to 1 that two particular persons do not occupy contiguous seats.

17. *P* bets Q £10 to £590 that three races will be won by the three horses *A*, *B*, *C*, against which the betting is 4 to 1, 3 to 1, and 2 to 1, respectively. The first race having been won by *A*, and it being known that the second race was won either by *B*, or by a horse *F* against which the betting was 6 to 1, find the value of *P*'s expectation.

Ans. £117. $5s.$ $5\frac{5}{11}d.$

18. Supposing the House of Commons composed of m Tories and n Whigs, find the probability that a Committee of $p+q$ selected by ballot will consist of p Tories and q Whigs.

Ans. $\frac{{}_mC_p \times {}_mC_q}{{}_{m+n}C_{p+q}}$[*].

19. There are 3 balls in a bag, of which one is white and one black, and the third white or black; determine the chance of drawing 2 black ones, if 2 be taken.

Ans. $\frac{1}{6}$.

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[*] ${}_nC_r$ signifies the Number of Combinations of n things taken r together.

20. A collection is made of ten letters taken at random from an alphabet consisting of 20 consonants and 5 vowels; what is the probability that it will contain 3 vowels and no more?

Ans. $\frac{60}{253}$.

21. At the game of whist, what is the chance of dealing one ace and no more to a specified person? And what is the chance of dealing one ace to each person?

(1) Ans. $\frac{32}{73}$ nearly. (2) Ans. $\frac{13182}{124950}$, or $\frac{1}{9}$ nearly.

22. There are two bags each containing 4 white and 4 black balls. Four are taken at a venture from one of them and transferred to the other. Then 8 being drawn from the latter, 6 of them prove white and two black; what is the chance that, if another be drawn, it will be white?

Ans. $\frac{9}{106}$.

23. At the game of whist what is the chance of the dealer and his partner holding the four honours?

Ans. $\frac{115}{1666}$, or $\frac{2}{29}$ nearly.

24. In dealing a pack of cards what is the chance that all the hearts will be found in the first 20 cards dealt, first without regard to order, and secondly in the order of their value?

(1) Ans. $\frac{8 \times \underline{|13} \times \underline{|39}}{\underline{|52}}$. (2) Ans. $\frac{8 \times \underline{|39}}{\underline{|52}}$.

25. A person puts his hand into a bag containing 12 balls, draws out a certain number at random, and transfers them without examination to a second empty bag. He then puts his hand into this second bag, and draws out a certain number in the same way as before. Shew that the odds in favour of his drawing out an odd number from the second bag are nearly 341 to 340.

26. Supposing it an even chance that, on *A* aged 46 marrying *B* aged 36, they will both be alive at the end of x years; find x, when of 86 persons born together one dies annually until all are extinct.

Ans. $x = 31{\cdot}62...$

27. A person 35 years of age wishes to buy an annuity for what may happen to remain of his life after 50 years of age. What is the Present Value of the annuity reckoning interest at 4 per cent. and using Dr. Halley's Table*?

Ans. $4\frac{1}{2}$ years' purchase nearly.

28. An annuity of £10, for the life of a person now 30 years old, is to commence at the end of 11 years, if another person now 40 should

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* See Page 273.

be then dead, or, if this should not be the case, at the end of any year beyond 11 years in which the former life remains, but not the latter. Required the Present Value of the annuity, reckoning interest at 4 per cent., and taking Dr. Halley's Table.

Ans. £34. 16*s*. in one single present payment; or £3 annual payment during the joint lives.

29. An estate or annuity of £10 for ever will be lost to the heirs of a person now 34, if his life should fail in 11 years. What ought he to give for the Assurance of it for this term according to Dr. Halley's Table?

Ans. £43·4.

30. A person now 40 is willing to pay £200 down, besides an annual payment for 10 years, to entitle him to a life-annuity of £44 after he attains the age of 50. What ought the annual payment to be, reckoning interest at 4 per cent. and using Dr. Halley's Table?

Ans. £8·55.

THE END.

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